# STA2053 Assignment 1

(1) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

2.11

$$AB = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = BA = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

(2) $AB = I \Rightarrow |AB| = |I| = 1$

2.14

$\Rightarrow |A|\,|B| = 1$, so $|A| \neq 0$, $|B| \neq 0$,
$A^{-1}$ exists and $B^{-1}$ exists

$AB = I \Rightarrow A^{-1}AB = A^{-1}I \Rightarrow B = A^{-1}$

$AB = I \Rightarrow ABB^{-1} = IB^{-1} \Rightarrow A = B^{-1}$

(3) $AB = I \Rightarrow \underbrace{CA}_{I}B = CI \Rightarrow B = C$

2.15

(4) $n \neq p$. Matrices must be square for inverses to exist.

2.16

(5) $AB(B^{-1}A^{-1}) = A\underbrace{BB^{-1}}A^{-1} = AA^{-1} = I$

2.17

So by problem 2.14, $(AB)^{-1} = B^{-1}A^{-1}$

(6) $A^{T}(A^{-1})^{T} = (A^{-1}A)^{T} = I^{T} = I$

2.18 So by problem 2.14 $(A^{T})^{-1} = (A^{-1})^{T}$

(7) Let $A$ be symmetric. $(A^{-1})^{T} = (A^{T})^{-1} = A^{-1}$ ✓

2.19

8 $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ so $a^T a = \sum_{i=1}^{n} a_i^2 \geq 0$

2.21

9 (a) Linearly dependent means $\exists\, x \neq 0$ with $Ax = 0$

2.27 Eigenvectors are non-zero, so $Ax = \lambda x = 0$ fits the definition of linear dependence

(b) Cols of A l.d. means $\exists\, x \neq 0$ with $Ax = 0$
Spse $A^{-1}$ exists. Then $A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$
Contradiction.

(c) Let the columns of A be linearly independent, and $(x, \lambda)$ be an eigenvector, eigenvalue pair. If $\lambda = 0$, then $Ax = 0 \Rightarrow x = 0$ by l.i. This is impossible, because $x$ is an eigenvector so $x \neq 0$. Therefore $\lambda \neq 0$

(d) $Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x \Rightarrow x = \lambda A^{-1}x$
$\Rightarrow A^{-1}x = \frac{1}{\lambda}x$. So eigenvalues are reciprocals, and eigenvectors are the same. Division by $\lambda$ is OK bec existence of $A^{-1}$ means cols of A are l.i. so $\lambda \neq 0$ by (c).

10 Let a have elements $= 0$ except for a one in
2.28 position $j$. Then $a^T \Sigma a = \sigma_{jj} > 0$.

11 Let $(\lambda, x)$ be an eigenvalue, eigenvector pair for $\Sigma$.
$\Sigma x = \lambda x \Rightarrow x^T \Sigma x = x^T \lambda x = \lambda x^T x = \lambda > 0$.

12 (a)

i) $D^{-1} = \begin{pmatrix} 1/\lambda_1 & & 0 \\ & 1/\lambda_2 & \\ 0 & & 1/\lambda_k \end{pmatrix}$

ii) $\Sigma C D^{-1} C^T = C D C^T C D^{-1} C^T = I$

(b) i) $D^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \sqrt{\lambda_k} \end{pmatrix}$

ii) $(C D^{1/2} C^T)^T = C^{TT} D^{\frac{1}{2}T} C^T = C D^{\frac{1}{2}} C^T = \Sigma^{1/2}$

iii) $\Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} = C D^{1/2} C^T C D^{\frac{1}{2}} C^T = C D C^T = \Sigma$

iv) $\Sigma^{\frac{1}{2}} x = 0 \Longrightarrow \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} x = \Sigma^{\frac{1}{2}} 0 = 0$

$\Longrightarrow \Sigma x = 0 \Longrightarrow x = 0$ by l.i. of cols of $\Sigma$.

(c) i) $\Sigma^{1/2} \Sigma^{-1/2} = C D^{1/2} C^T C D^{-1/2} C^T = I$

So $\Sigma^{-1/2}$ is the inverse of $\Sigma^{1/2}$ by problem 2.

ii) $\Sigma^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} = C D^{-\frac{1}{2}} C^T C D^{-\frac{1}{2}} C^T$

$= C D^{-\frac{1}{2}} D^{-\frac{1}{2}} C^T$

$= C D^{-1} C^T = \Sigma^{-1}$

(13) (a) $\Sigma x = 0 \Rightarrow \Sigma^{-1}\Sigma x = \Sigma^{-1} 0 = 0 \Rightarrow x = 0$ ✓

(b) Let $(\lambda, x)$ be any eigenvalue, eigenvector pair for $\Sigma$. $\Sigma x = \lambda x \Rightarrow x^T \Sigma x = x^T \lambda x = \lambda x^T x$ $= \lambda \geq 0$. So by 12b, $\Sigma^{1/2}$ exists.

Let $a \neq 0$. Because $a^T \Sigma a \geq 0$, either $a^T \Sigma a > 0$ or $a^T \Sigma a = 0$. Suppose $a^T \Sigma a = 0$

Then $0 = \underbrace{a^T \Sigma^{\frac{1}{2}}}_{z^T} \underbrace{\Sigma^{1/2} a}_{z} = z^T z = 0 \Rightarrow z = 0$

$\Sigma^{1/2} a = 0$ for $a \neq 0$, but cols of $\Sigma^{\frac{1}{2}}$ are linearly independent by 12biv because two cols of $\Sigma$ are l.i. Therefore $a = 0$, a contradiction. Thus $a^T \Sigma a = 0$ is impossible, and we have $a^T \Sigma a > 0$. That is, $\Sigma$ is positive definite. ∎

(14) Want to apply Problem 13 to conclude $\Sigma^{-1}$ p.d.. Need

- $\Sigma^{-1}$ symmetric. Follows from Problem 7.
- $\Sigma^{-1}$ has an inverse. Of course.
- $\Sigma^{-1}$ non-negative definite

By Problem 11, the eigenvalues of $\Sigma$ are positive. Since the eigenvalues of $\Sigma^{-1}$ are the reciprocals by problem 9d, they are also positive. So by Problem 12b, $\Sigma^{-\frac{1}{2}}$ exists.

Let $a \neq 0$ $a^T \Sigma^{-1} a = \underbrace{a^T \Sigma^{-1/2}}_{z^T} \underbrace{\Sigma^{-\frac{1}{2}} a}_{z} = z^T z \geq 0$, and

the conclusion follows from problem 13b. ∎