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Sample Questions: Maximum Likelihood Part 2

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1. Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables.

   (a) Derive formulas for the maximum likelihood estimates of μ and σ^2. We will establish that it's a maximum later. Show your work and **circle your final answer**.

$$\ell(\mu, \sigma^2) = \log \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \log\left((\sigma^2)^{-\frac{n}{2}} 2\pi^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2} \right)$$

$$= -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial \ell}{\partial \mu} = 0+0 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{\partial}{\partial\mu}(x_i-\mu)^2$$

$$= \frac{+1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(x_i-\mu)(+1) = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n}x_i - n\mu\right) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n}x_i = n\mu \Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n}x_i}{n} = \bar{x}$$

$$\frac{\partial \ell}{\partial \sigma^2} = \frac{\partial}{\partial\sigma^2}\left(-\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2}\cdot(\sigma^2)^{-1} \right)$$

$$= \frac{-n}{2\sigma^2} + 0 - \frac{\sum_{i=1}^{n}(x_i-\mu)}{2}(+1)(\sigma^2)^{-2}$$

$$= \frac{-n}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^4}$$

Set $\mu = \bar{x}$ and set $=0$

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$$\Rightarrow \frac{n}{\cancel{2}\cancel{\sigma}^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\cancel{2}\cancel{\sigma}^2\sigma^2}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$$

So $(\hat{\mu}, \hat{\sigma}^2) = \left(\bar{x}, \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}\right)$

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(b) Calculate the Hessian of the minus log likelihood function: $\mathbf{H} = \left[\frac{\partial^2(-\ell)}{\partial\theta_i\partial\theta_j}\right]$. Show your work.

$$-\frac{\partial^2 \ell}{\partial \mu^2} = -\frac{\partial}{\partial \mu}\left(\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right)\right)$$

$$= (-1)\frac{-n}{\sigma^2} = \frac{n}{\sigma^2}$$

$$-\frac{\partial^2 \ell}{\partial \mu\, \partial \sigma^2} = -\frac{\partial}{\partial \sigma^2}\left(\left(\sum x_i - n\mu\right)\left(\sigma^2\right)^{-1}\right)$$

$$= (+1)\left(n\bar{x} - n\mu\right)(+1)\left(\sigma^2\right)^{-2}$$

$$= \frac{n(\bar{x}-\mu)}{\sigma^4}$$

$$-\frac{\partial^2 \ell}{\partial (\sigma^2)^2} = -\frac{\partial}{\partial \sigma^2}\left(-\frac{n}{2}\left(\sigma^2\right)^{-1} + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\left(\sigma^2\right)^{-2}\right)$$

$$= \frac{n}{2}(-1)\left(\sigma^2\right)^{-2} + \frac{1}{\cancel{2}}\sum_{i=1}^{n}(x_i-\mu)^2(+\cancel{2})\left(\sigma^2\right)^{-3}$$

$$= \frac{-n}{2\sigma^4} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^6}$$

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$H(\hat{\theta})$

$$= \begin{pmatrix} \dfrac{n}{\sigma^2} & \dfrac{n(\bar{x}-\bar{x})}{\sigma^4} \\ 0 & \dfrac{-n}{2\hat{\sigma}^4} + \dfrac{n\,\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}{\hat{\sigma}^6} \end{pmatrix}$$

$\hat{\sigma}^2$

$$= \begin{pmatrix} \dfrac{n}{\hat{\sigma}^2} & 0 \\ 0 & \dfrac{-n}{2\hat{\sigma}^4} + \dfrac{n\,\cancel{\hat{\sigma}^2}}{\cancel{\hat{\sigma}^2}\,\hat{\sigma}^4} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{n}{\hat{\sigma}^2} & 0 \\ 0 & \dfrac{n}{2\hat{\sigma}^4} \end{pmatrix}$$

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(c) Give $\widehat{\mathbf{V}}_n$, the estimated asymptotic variance-covariance matrix of the MLE. Show some work.

$$\hat{V}_n = \left(H(\hat{\theta})\right)^{-1}$$

$$= \begin{pmatrix} \frac{n}{\hat{\sigma}^2} & 0 \\ 0 & \frac{n}{2\hat{\sigma}^4} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{\hat{\sigma}^2}{n} & 0 \\ 0 & \frac{2\hat{\sigma}^4}{n} \end{pmatrix}$$

$$\approx \operatorname{cov}\begin{pmatrix} \bar{x} \\ \hat{\sigma}^2 \end{pmatrix}$$

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(d) Consider a large-sample Z-test of $H_0 : \mu = \mu_0$. Give an explicit formula for the test statistic. This is something you would be able to compute with a calculator given $\widehat{\mu}$ and $\widehat{\sigma}^2$.

Estimated asymptotic variance of $\hat{\mu} = \bar{x}$ is $\frac{\hat{\sigma}^2}{n}$, SE is $\hat{\sigma}/\sqrt{n}$

$$Z = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\hat{\sigma}}$$

(e) Consider a large-sample Z-test of $H_0 : \sigma^2 = \sigma_0^2$. Give an explicit formula for the test statistic. This is something you would be able to compute with a calculator.

Est. variance of $\hat{\sigma}^2$ is $\frac{2\hat{\sigma}^4}{n}$, so

$$Z = \frac{\hat{\sigma}_n^2 - \sigma_0^2}{\sqrt{\frac{2\hat{\sigma}^4}{n}}} = \frac{\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2)}{\sqrt{2}\,\hat{\sigma}^2}$$

$$G^2 = -2\log\frac{L(\hat{\theta}_0)}{L(\hat{\theta})}$$

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(f) Consider the large-sample likelihood ratio test of $H_0 : \mu = \mu_0$. Derive an explicit formula for the test statistic G^2. Show your work and *keep simplifying!*.

$$L(\mu_0,\sigma^2) = (\sigma^2)^{-\frac{n}{2}}(2\pi)^{-\frac{n}{2}}\, Exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2\right\}$$

$$\ell(\mu_0,\sigma^2) = -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2(\sigma^2)^{-1}$$

$$\ell' = \frac{-n}{2\sigma^2} - 0 + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2(+1)(\sigma^2)^{-2} \overset{set}{=} 0$$

$$\Rightarrow \frac{n}{\cancel{2\sigma^2}} = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\cancel{2\sigma^2}\sigma^2} \Rightarrow \hat{\sigma}_0^2 = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n}$$

$$G^2 = -2\log\frac{(\hat{\sigma}_0^2)^{-n/2}\cancel{(2\pi)^{-n/2}}\, Exp\left\{-\frac{n}{2\cancel{\hat{\sigma}_0^2}}\cancel{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2}\right\}}{(\hat{\sigma}^2)^{-n/2}\cancel{(2\pi)^{-n/2}}\, Exp\left\{-\frac{n}{2\cancel{\hat{\sigma}^2}}\cancel{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}\right\}}$$

$$-2\log\left(\left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2}\cdot\frac{\cancel{e^{-n/2}}}{\cancel{e^{-n/2}}}\right)$$

$$= -\cancel{2}\left(\frac{n}{\cancel{2}}\log\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right) = -n(\log\hat{\sigma}^2 - \log\hat{\sigma}_0^2)$$

$$= n(\log\hat{\sigma}_0^2 - \log\hat{\sigma}^2)$$

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2g(i) Let $a > 0$, $X \sim N(\mu, \sigma^2)$, $Y = aX$

$E(Y) = a\mu$ $\quad var(Y) = Var(aX) = a^2 var(X)$

$= a^2\sigma^2$

$$CV_y = \frac{SD(Y)}{E(Y)} = \frac{\sqrt{a^2\sigma^2}}{a\mu} = \frac{\cancel{a}\sigma}{\cancel{a}\mu} = CV_x$$

$g(\mu, \sigma^2) = \frac{\sigma}{\mu}$

(iii) $g(\hat{\theta}) \dot{\sim} N(g(\theta), \dot{g}(\theta) V_\theta \dot{g}(\theta)^T)$

def

$$\dot{g}(\theta) = \left(\frac{\partial g}{\partial \theta_1}, \frac{\partial g}{\partial \theta_2} \cdots \frac{\partial g}{\partial \theta_k}\right)$$

$$\dot{g}(\mu, \sigma^2) = \left(\frac{\partial}{\partial \mu}\frac{\sigma}{\mu}, \frac{\partial}{\partial \sigma^2}\frac{\sigma}{\mu}\right)$$

$$= \left(\frac{\partial}{\partial \mu}\sigma\mu^{-1}, \frac{\partial}{\partial \sigma^2}\frac{1}{\mu}(\sigma^2)^{\frac{1}{2}}\right)$$

$$= \left(\sigma(-1)\mu^{-2}, \frac{1}{\mu}\frac{1}{2}(\sigma^2)^{-\frac{1}{2}}\right)$$

$$= \left(\frac{-\sigma}{\mu^2}, \frac{1}{2\mu\sigma}\right)$$

(2g i) $CV = \frac{\sigma}{\mu}$

Let $X \sim ?(\mu, \sigma^2)$, $Y = aX$

$E(Y) = a\mu$, $Var(Y) = Var(aX) = a^2 Var(X) = a^2 \sigma^2$

So $CV_Y = \frac{\sqrt{a^2\sigma^2}}{a\mu} = \frac{a\sigma}{a\mu} = \frac{\sigma}{\mu} = CV$

(iii) $se = \sqrt{\dot{g}(\hat{\theta})\, \hat{V}_n\, \dot{g}(\hat{\theta})^T}$

$\dot{g}(\mu, \sigma^2) = \left(\frac{\partial}{\partial\mu} \sigma \mu^{-1}, \frac{\partial}{\partial\sigma^2} \frac{1}{\mu} (\sigma^2)^{\frac{1}{2}} \right)$

$= \left(\sigma (-1) \mu^{-2}, \frac{1}{\mu} \frac{1}{2} (\sigma^2)^{-\frac{1}{2}} \right)$

$= \left(\frac{-\sigma}{\mu^2}, \frac{1}{2\mu\sigma} \right)$