

## STA 312f23 Formulas

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

$$E(X) \stackrel{def}{=} \sum_x x p_x(x) \text{ or } \int_{-\infty}^\infty x f_x(x) dx$$

$$Var(X) \stackrel{def}{=} E((X - \mu)^2)$$

$$L(\theta) = \prod_{i=1}^n p(y_i|\theta)$$

$$L(\theta) = \prod_{i=1}^n f(t_i|\theta)^{\delta_i} S(t_i|\theta)^{1-\delta_i}$$

$$\hat{\theta}_n \sim N(\theta, \frac{1}{nI(\theta)})$$

$$\hat{v}_n = 1 / -\ell''(\hat{\theta})$$

$$95\% \text{ CI: } \hat{\theta} \pm 1.96 \times S_{\hat{\theta}}$$

If  $g: \mathbb{R} \rightarrow \mathbb{R}$

$$\hat{\theta}_n \sim N_k(\theta, \frac{1}{n} \mathcal{I}(\theta)^{-1})$$

$$\mathbf{H}(\theta) = \left[ -\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell(\theta) \right]$$

If  $g: \mathbb{R}^k \rightarrow \mathbb{R}$

$$\dot{g}(\theta) = \left( \frac{\partial g}{\partial \theta_1}, \dots, \frac{\partial g}{\partial \theta_k} \right)$$

$$G^2 = -2 \log \left( \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta} L(\theta)} \right) = -2 \log \left( \frac{L(\hat{\theta}_0)}{L(\hat{\theta})} \right)$$

$$S(t) \stackrel{def}{=} P(T > t) = 1 - F(t)$$

$$h(t) = \frac{f(t)}{S(t)}$$

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \text{ and } \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$E(g(X)) = \sum_x g(x) p_x(x) \text{ or } \int_{-\infty}^\infty g(x) f_x(x) dx$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$L(\theta) = \prod_{i=1}^n f(y_i|\theta) \quad \ell(\theta) = \log L(\theta)$$

$$\ell(\theta) = \sum_{i=1}^n \delta_i \log f(t_i|\theta) + \sum_{i=1}^n (1 - \delta_i) \log S(t_i|\theta)$$

$$I(\theta) = -E \frac{\partial^2}{\partial \theta^2} \log f(X|\theta)$$

$$S_{\hat{\theta}} = \sqrt{\hat{v}_n}$$

$$Z_n = \frac{\hat{\theta} - \theta_0}{S_{\hat{\theta}}}$$

$$g(\hat{\theta}) \sim N(g(\theta), g'(\theta)^2 v_n)$$

$$\mathcal{I}(\theta) = \left[ -E \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(Y; \theta) \right) \right]$$

$$\hat{\mathbf{V}}_n = \mathbf{H}(\hat{\theta})^{-1} \text{ estimates } \frac{1}{n} \mathcal{I}(\theta)^{-1}$$

$$g(\hat{\theta}) \sim N(g(\theta), \dot{g}(\theta) \mathbf{V}_n \dot{g}(\theta)^\top)$$

$$W_n = (\mathbf{L}\hat{\theta}_n - \mathbf{h})^\top \left( \mathbf{L}\hat{\mathbf{V}}_n \mathbf{L}^\top \right)^{-1} (\mathbf{L}\hat{\theta}_n - \mathbf{h})$$

$$h(t) \stackrel{def}{=} \lim_{\Delta \rightarrow 0} \frac{P(t < T < t + \Delta | T > t)}{\Delta}, \text{ where } \Delta > 0$$

$$S(t) = \exp\left\{-\int_0^t h(x) dx\right\}$$

Distribution	Density	S(t)	E(T)	Median
Exponential	$f(t \lambda) = \lambda e^{-\lambda t} I(t \geq 0)$	$e^{-\lambda t}$	$\frac{1}{\lambda}$	$\frac{\log(2)}{\lambda}$
Weibull	$f(t \alpha, \lambda) = \alpha \lambda (\lambda t)^{\alpha-1} \exp\{-(\lambda t)^\alpha\} I(t \geq 0)$	$e^{-(\lambda t)^\alpha}$	$\frac{\Gamma(1+1/\alpha)}{\lambda}$	$\frac{\log(2)^{1/\alpha}}{\lambda}$
Gumbel $G(\mu, \sigma)$	$f(y \mu, \sigma) = \frac{1}{\sigma} \exp\left\{\left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)}\right\}$	$e^{-e^{\left(\frac{t-\mu}{\sigma}\right)}}$	$\sigma \Gamma'(1) + \mu$	$\sigma \log(\log(2)) + \mu$
Log-normal $(\mu, \sigma^2)$	$f(y \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{(\log(t)-\mu)^2}{2\sigma^2}\right\} \frac{1}{t} I(t > 0)$		$\exp\left(\mu + \frac{\sigma^2}{2}\right)$	$e^\mu$

$T \sim \text{Lognormal}(\mu, \sigma^2)$  if and only if  $X = \log(T) \sim N(\mu, \sigma^2)$

Log of standard exponential is Gumbel(0,1), also called the standard extreme value distribution.

If  $Z \sim G(0, 1)$ , MGF is  $M_z(t) = \Gamma(t + 1)$ ,  $E(Z) = \Gamma'(1)$ ,  $Var(Z) = \frac{\pi^2}{6}$ , and  $Y = \sigma Z + \mu \sim G(\mu, \sigma)$ .

Log of Weibull with  $\alpha = 1/\sigma$  and  $\lambda = e^{-\mu}$  is Gumbel( $\mu, \sigma$ ).

Kaplan-Meier estimate: Discrete time.

- $p_j$  = the probability of surviving past time  $t_j$ , given survival to time  $t_{j-1}$ .  
 $S(t_k) = \prod_{j=1}^k p_j$ .
- $d_j$  is the number of deaths at time  $t_j$ , and  $n_j$  is the number of individuals at risk before time  $t_j$ .
- $\hat{p}_j = \frac{n_j - d_j}{n_j}$ ,  $\hat{S}(t_k) = \prod_{j=1}^k \hat{p}_j$ ,  $\hat{S}(t) = \prod_{t_j \leq t} \hat{p}_j$ .
- $\hat{S}(t) \sim N\left(S(t), S(t)^2 \sum_{t_j \leq t} \frac{1-p_j}{n_j p_j}\right)$ .
- The standard error of  $\hat{S}(t)$  is  $\hat{S}(t) \sqrt{\sum_{t_j \leq t} \left(\frac{d_j}{n_j(n_j - d_j)}\right)}$ .

Weibull Regression:  $t_i = \exp\{\beta_0 + \beta_1 x_{i,1} + \dots + \beta_{p-1} x_{i,p-1}\} \cdot \epsilon_i^\sigma = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \epsilon_i^\sigma$ , where  $\epsilon_1 \sim \exp(1)$ .

- $t_i \sim \text{Weibull}$ , with  $\alpha = 1/\sigma$  and  $\lambda = e^{-\mathbf{x}_i^\top \boldsymbol{\beta}}$ .
- $E(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \Gamma(\sigma + 1)$ ,  $\text{Median}(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \log(2)^\sigma$ ,  $h_i(t) = \frac{1}{\sigma} \exp\{-\frac{1}{\sigma} \mathbf{x}_i^\top \boldsymbol{\beta}\} t^{\frac{1}{\sigma}-1}$ .
- $S(t) = \exp\left\{-e^{-\frac{1}{\sigma} \mathbf{x}_i^\top \boldsymbol{\beta}} t^{\frac{1}{\sigma}}\right\}$

Log-normal Regression Regression:  $t_i = \exp\{\beta_0 + \beta_1 x_{i,1} + \dots + \beta_{p-1} x_{i,p-1}\} \cdot \epsilon_i^\sigma = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \epsilon_i^\sigma$ , where  $\epsilon_1 \sim \text{Log-normal}(0,1)$ .

- $t_i \sim \text{Log-normal}(\mu, \sigma^2)$ , with  $\mu = e^{\mathbf{x}_i^\top \boldsymbol{\beta}}$ .
- $E(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta} + \frac{1}{2}\sigma^2}$ ,  $\text{Median}(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta}}$ .

Proportional Hazards Regression

- $h(t) = h_0(t) e^{\mathbf{x}^\top \boldsymbol{\beta}}$ .
- $\text{PL}(\boldsymbol{\beta}) = \prod_{i=1}^D \left( \frac{e^{\mathbf{x}_{(i)}^\top \boldsymbol{\beta}}}{\sum_{j \in R_{(i)}} e^{\mathbf{x}_j^\top \boldsymbol{\beta}}} \right)$ .

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<http://www.utstat.toronto.edu/brunner/oldclass/312f23>