STA 312f23 Formulas

$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt$

$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$E(X) \stackrel{def}{=} \sum_x x\, p_x(x)$ or $\int_{-\infty}^\infty x\, f_x(x)\, dx$

$E(g(X)) = \sum_x g(x)\, p_x(x)$ or $\int_{-\infty}^\infty g(x)\, f_x(x)\, dx$

$Var(X) \stackrel{def}{=} E\left((X-\mu)^2\right)$

$Var(X) = E(X^2) - [E(X)]^2$

$L(\theta) = \prod_{i=1}^n p(y_i|\theta)$

$L(\theta) = \prod_{i=1}^n f(y_i|\theta) \qquad \ell(\theta) = \log L(\theta)$

$L(\theta) = \prod_{i=1}^n f(t_i|\theta)^{\delta_i}\, S(t_i|\theta)^{1-\delta_i}$

$\ell(\theta) = \sum_{i=1}^n \delta_i \log f(t_i|\theta) + \sum_{i=1}^n (1-\delta_i) \log S(t_i|\theta)$

$\widehat{\theta}_n \stackrel{\cdot}{\sim} N(\theta, \frac{1}{n\, I(\theta)})$

$I(\theta) = -E \frac{\partial^2}{\partial\theta^2} \log f(X|\theta)$

$\widehat{v}_n = 1/-\ell''(\widehat{\theta})$

$S_{\widehat{\theta}} = \sqrt{\widehat{v}_n}$

95% CI: $\widehat{\theta} \pm 1.96 \times S_{\widehat{\theta}}$

$Z_n = \frac{\widehat{\theta}-\theta_0}{S_{\widehat{\theta}}}$

If $g : \mathbb{R} \rightarrow \mathbb{R}$

$g(\widehat{\theta}) \stackrel{\cdot}{\sim} N\left(g(\theta), g'(\theta)^2\, v_n\right)$

$\widehat{\boldsymbol{\theta}}_n \stackrel{\cdot}{\sim} N_k\left(\boldsymbol{\theta}, \frac{1}{n}\boldsymbol{\mathcal{I}}(\boldsymbol{\theta})^{-1}\right)$

$\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}) = \left[-E\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j} \log f(Y;\boldsymbol{\theta})\right)\right]$

$\mathbf{H}(\boldsymbol{\theta}) = \left[-\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ell(\boldsymbol{\theta})\right]$

$\widehat{\mathbf{V}}_n = \mathbf{H}(\widehat{\boldsymbol{\theta}})^{-1}$ estimates $\frac{1}{n}\boldsymbol{\mathcal{I}}(\boldsymbol{\theta})^{-1}$

If $g : \mathbb{R}^k \rightarrow \mathbb{R}$

$\dot{\mathrm{g}}(\boldsymbol{\theta}) = \left(\frac{\partial g}{\partial\theta_1}, \ldots, \frac{\partial g}{\partial\theta_k}\right)$

$g(\widehat{\boldsymbol{\theta}}) \stackrel{\cdot}{\sim} N\left(g(\boldsymbol{\theta}), \dot{\mathrm{g}}(\boldsymbol{\theta})\mathbf{V}_n\, \dot{\mathrm{g}}(\boldsymbol{\theta})^\top\right)$

$G^2 = -2\log\left(\frac{\max_{\boldsymbol{\theta}\in\Theta_0} L(\boldsymbol{\theta})}{\max_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta})}\right) = -2\log\left(\frac{L(\widehat{\boldsymbol{\theta}}_0)}{L(\widehat{\boldsymbol{\theta}})}\right)$

$W_n = (\mathbf{L}\widehat{\boldsymbol{\theta}}_n - \mathbf{h})^\top \left(\mathbf{L}\widehat{\mathbf{V}}_n\mathbf{L}^\top\right)^{-1} (\mathbf{L}\widehat{\boldsymbol{\theta}}_n - \mathbf{h})$

$S(t) \stackrel{def}{=} P(T > t) = 1 - F(t)$

$h(t) \stackrel{def}{=} \lim_{\Delta\rightarrow 0} \frac{P(t<T<t+\Delta|T>t)}{\Delta}$, where $\Delta > 0$

$h(t) = \frac{f(t)}{S(t)}$

$S(t) = \exp\{-\int_0^t h(x)\, dx\}$

Distribution	Density	$\mathbf{S(t)}$	$\mathbf{E(T)}$	Median
Exponential	$f(t\|\lambda) = \lambda e^{-\lambda t}\, I(t \geq 0)$	$e^{-\lambda t}$	$\frac{1}{\lambda}$	$\frac{\log(2)}{\lambda}$
Weibull	$f(t\|\alpha,\lambda) = \alpha\lambda(\lambda t)^{\alpha-1}\, \exp\{-(\lambda t)^\alpha\}\, I(t \geq 0)$	$e^{-(\lambda t)^\alpha}$	$\frac{\Gamma(1+1/\alpha)}{\lambda}$	$\frac{\log(2)^{1/\alpha}}{\lambda}$
Gumbel $G(\mu,\sigma)$	$f(y\|\mu,\sigma) = \frac{1}{\sigma}\, \exp\left\{\left(\frac{y-\mu}{\sigma}\right) - e^{\left(\frac{y-\mu}{\sigma}\right)}\right\}$	$e^{-e^{\left(\frac{t-\mu}{\sigma}\right)}}$	$\sigma\Gamma'(1) + \mu$	$\sigma\log(\log(2)) + \mu$
Log-normal(μ,σ^2)	$f(y\|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, \exp - \left\{\frac{(\log(t)-\mu)^2}{2\sigma^2}\right\}\, \frac{1}{t}\, I(t > 0)$		$\exp\left(\mu + \frac{\sigma^2}{2}\right)$	e^μ

$T \sim$ Lognormal(μ,σ^2) if and only if $X = \log(T) \sim N(\mu,\sigma^2)$

Log of standard exponential is Gumbel(0,1), also called the standard extreme value distribution.
If $Z \sim G(0,1)$, MGF is $M_z(t) = \Gamma(t+1)$, $E(Z) = \Gamma'(1)$, $Var(Z) = \frac{\pi^2}{6}$, and $Y = \sigma Z + \mu \sim G(\mu,\sigma)$.
Log of Weibull with $\alpha = 1/\sigma$ and $\lambda = e^{-\mu}$ is Gumbel(μ,σ).

Kaplan-Meier estimate: Discrete time.

- p_j = the probability of surviving past time t_j, given survival to time t_{j-1}. $S(t_k) = \prod_{j=1}^{k} p_j$.
- d_j is the number of deaths at time t_j, and n_j is the number of individuals at risk before time t_j.
- $\widehat{p}_j = \frac{n_j - d_j}{n_j}$, $\quad \widehat{S}(t_k) = \prod_{j=1}^{k} \widehat{p}_j$, $\quad \widehat{S}(t) = \prod_{t_j \leq t} \widehat{p}_j$.
- $\widehat{S}(t) \stackrel{.}{\sim} N\left(S(t), S(t)^2 \sum_{t_j \leq t} \frac{1-p_j}{n_j p_j}\right)$.
- The standard error of $\widehat{S}(t)$ is $\widehat{S}(t)\sqrt{\sum_{t_j \leq t}\left(\frac{d_j}{n_j(n_j-d_j)}\right)}$.

Weibull Regression: $t_i = \exp\{\beta_0 + \beta_1 x_{i,1} + \ldots + \beta_{p-1} x_{i,p-1}\} \cdot \epsilon_i^\sigma = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \epsilon_i^\sigma$, where $\epsilon_1 \sim \exp(1)$.

- $t_i \sim$ Weibull, with $\alpha = 1/\sigma$ and $\lambda = e^{-\mathbf{x}_i^\top \boldsymbol{\beta}}$.
- $E(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta}}\, \Gamma(\sigma+1)$, Median$(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \log(2)^\sigma$, $h_i(t) = \frac{1}{\sigma} \exp\{-\frac{1}{\sigma}\mathbf{x}_i^\top \boldsymbol{\beta}\} t^{\frac{1}{\sigma}-1}$.
- $S(t) = \exp\left\{-e^{-\frac{1}{\sigma}\mathbf{x}_i^\top \boldsymbol{\beta}} t^{\frac{1}{\sigma}}\right\}$

Log-normal Regression Regression: $t_i = \exp\{\beta_0 + \beta_1 x_{i,1} + \ldots + \beta_{p-1} x_{i,p-1}\} \cdot \epsilon_i^\sigma = e^{\mathbf{x}_i^\top \boldsymbol{\beta}} \epsilon_i^\sigma$, where $\epsilon_1 \sim$ Log-normal(0,1).

- $t_i \sim$ Log-normal(μ, σ^2), with $\mu = e^{\mathbf{x}_i^\top \boldsymbol{\beta}}$.
- $E(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta} + \frac{1}{2}\sigma^2}$, Median$(t_i) = e^{\mathbf{x}_i^\top \boldsymbol{\beta}}$.

Proportional Hazards Regression

- $h(t) = h_0(t)\, e^{\mathbf{x}^\top \boldsymbol{\beta}}$.
- $$\text{PL}(\boldsymbol{\beta}) = \prod_{i=1}^{D} \left(\frac{e^{\mathbf{x}_{(i)}^\top \boldsymbol{\beta}}}{\displaystyle\sum_{j \in R_{(i)}} e^{\mathbf{x}_j^\top \boldsymbol{\beta}}} \right).$$
