Assignment 6

(1) $$P(T > t_j \mid T > t_{j-1}) = \frac{P(T > t_j \cap T > t_{j-1})}{P(T > t_{j-1})}$$

$$= \frac{P(T > t_j)}{P(T > t_{j-1})} = \frac{S(t_j)}{S(t_{j-1})}$$

(2) $$\prod_{j=1}^{k} p_j = \prod_{j=1}^{k} \frac{S(t_j)}{S(t_{j-1})}$$

$$= \frac{S(\cancel{t_1})}{\underbrace{S(t_0)}_{=1}} \cdot \frac{S(\cancel{t_2})}{S(\cancel{t_1})} \cdot \frac{S(\cancel{t_3})}{S(\cancel{t_2})} \cdots \frac{S(\cancel{t_{k-1}})}{S(\cancel{t_{k-2}})} \cdot \frac{S(t_k)}{S(\cancel{t_{k-1}})}$$

$$= S(t_k)$$

(3) (a) $E(\hat{p}) = p$, $Var(\hat{p}) = \frac{p(1-p)}{n}$ $\left(= \frac{\sigma^2}{n} \right)$

(b) $\hat{p} \dot{\sim} N\left(p, \frac{p(1-p)}{n}\right)$

(4) $\hat{p}_j = 1 - \frac{d_j}{n_j}$ (one minus conditional probability of death)

$= \frac{n_j - d_j}{n_j}$ The proportion who lived

~~(5) (a) $\hat{p} \dot{\sim} N\left(p_j, \frac{p_j(1-p_j)}{n_j}\right)$~~

(5) (a) $\hat{p}_j \dot{\sim} N\left(p_j, \frac{p_j(1-p_j)}{n_j}\right)$

(5b) Using the one-variable delta method, $\hat{P}_j$ should be asymptotically normal with expected value P_j. To get the variance, with $g(x) = \log x$

$g'(x) = \frac{d}{dx} \log x = \frac{1}{x}$, so asymptotic variance is

$\frac{1}{P_j^2} \frac{P_j(1-P_j)}{n_j} = \frac{1-P_j}{n_j P_j}$, and (loosely)

$$\log \hat{P}_j \mathrel{\dot\sim} N\left(\log P_j, \frac{1-P_j}{n_j P_j}\right)$$

(6) (a) $\log \hat{S}(t) = \sum_{t_j \le t} \log \hat{P}_j$

(b) $E(\log \hat{S}(t)) \approx \sum_{t_j \le t} E(\log \hat{P}_j)$

$= \sum_{t_j \le t} \log P_j$.

(c) $Var(\log \hat{S}(t)) \approx \sum_{t_j \le t} Var(\log \hat{P}_j)$

$= \sum_{t_j \le t} \frac{1-P_j}{n_j P_j}$, so we guess

(d) $\log \hat{S}(t) \mathrel{\dot\sim} N\left(\overbrace{\sum_{t_j \le t} \log P_j}^{\text{or } \log(S(t))}, \sum_{t_j \le t} \frac{1-P_j}{n_j P_j}\right)$

(7) Have asymptotic distribution of $\log \hat{S}(t)$. Want distribution of $g(\log \hat{S}(t)) = \operatorname{Exp}\{\log \hat{S}(t)\}$. Using one-variable delta method $g'(x) = g(x)$, so

$$\hat{S}(t) \,\dot{\sim}\, N\left(S(t),\ S(t)^2 \sum_{t_j \le t} \frac{1-p_j}{n_j p_j}\right)$$

(8) Standard error is the square root of the estimated variance. Estimated variance is

$$\hat{S}(t)^2 \sum_{t_j \le t} \frac{1-\hat{p}_j}{n_j \hat{p}_j} = \hat{S}(t)^2 \sum_{t_j \le t} \frac{d_j / n_j}{\cancel{n_j} \frac{n_j - d_j}{\cancel{n_j}}}$$

$$= \hat{S}^2(t) \sum_{t_j \le t} \frac{d_j}{n_j(n_j - d_j)}$$, and standard error is

$$\hat{S}(t) \sqrt{\sum_{t_j \le t} \frac{d_j}{n_j(n_j - d_j)}}$$, where

$$\hat{S}(t) = \prod_{t_j \le t} \hat{p}_j = \prod_{t_j \le t} \left(1 - \frac{d_j}{n_j}\right)$$

(9)

t_i	n_i	d_i	$\hat{p}_i$	$\hat{S}(t_i)$
0	100	0	1	1
2	100	15	85/100	0.85
4	83	5	78/83	0.7988
5	73	10	63/73	0.6894

(10) 5 observations were censored between $t=4$ & $t=5$

(12) (a) For an exponential distribution get median by setting $S(t) = \frac{1}{2}$, so $e^{-\lambda t} = \frac{1}{2} \Rightarrow -\lambda t = \log \frac{1}{2}$

$$-\lambda t = \log(2^{-1}) = -\log 2 \Rightarrow \lambda t = \log 2 \Rightarrow t = \frac{\log 2}{\lambda}$$

And MLE of median is $\frac{\log 2}{\hat{\lambda}}$

To get a confidence interval for the median, either use the delta method or transform the confidence interval for λ.

- Delta method. From Question 1b of assignment 5,

$$\mathrm{Var}(\hat{\lambda}) \approx \frac{\lambda^2}{\sum_{i=1}^n \delta_i} = v_n \quad . \quad g(x) = (\log 2)\, x^{-1}$$

$$g'(x) = -(\log 2)\, x^{-2} = -\frac{\log 2}{x^2}$$, so asymptotic variance of median is $g'(\lambda)^2 v_n = \frac{(\log 2)^2}{\lambda^4} \frac{\lambda^2}{\sum \delta_i}$

$$= \frac{(\log 2)^2}{\lambda^2 \sum_{i=1}^n \delta_i}$$, and standard error is

$$se_{med} = \frac{\log 2}{\hat{\lambda}\sqrt{\sum \delta_i}}$$

$$se_{med} = \frac{\log 2}{\hat{\lambda}\sqrt{\sum \delta_i}}$$

- Transform confidence interval.

$$0.95 \approx P(A < \lambda < B) = P\left(\frac{1}{A} > \frac{1}{\lambda} > \frac{1}{B}\right)$$

$$= P\left(\frac{\log 2}{B} < \frac{\log 2}{\lambda} = Med < \frac{\log 2}{A}\right)$$

```
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[R.app GUI 1.79 (8160) x86_64-apple-darwin17.0]

[Workspace restored from /Users/brunner/.RData]
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> # A6
>
> rm(list=ls()); options(scipen=999)
> exdata = read.table("http://www.utstat.utoronto.ca/brunner/data/legal/expo.data2.txt")
> head(exdata); Time = exdata$Time; Uncensored = exdata$Uncensored
   Time Uncensored
1 0.179          0
2 1.024          1
3 0.189          1
4 0.345          1
5 0.977          1
6 0.241          1
>
> # 11(a)
> # install.packages("survival",dependencies=TRUE) # Only need to do this once
> library(survival)
> y = Surv(Time,Uncensored)
> km = survfit(y ~ 1); km
Call: survfit(formula = y ~ 1)

      n events median 0.95LCL 0.95UCL
[1,] 50     40  0.351   0.284   0.758
> sumkm = summary(km); sumkm
Call: survfit(formula = y ~ 1)

  time n.risk n.event survival std.err lower 95% CI upper 95% CI
 0.026     50       2   0.9600  0.0277      0.90719        1.000
 0.032     47       1   0.9396  0.0338      0.87557        1.000
 0.058     44       1   0.9182  0.0392      0.84448        0.998
 0.062     43       1   0.8969  0.0437      0.81511        0.987
 0.100     41       1   0.8750  0.0478      0.78610        0.974
 0.101     40       1   0.8531  0.0514      0.75811        0.960
 0.109     39       1   0.8312  0.0545      0.73095        0.945
 0.117     38       1   0.8094  0.0573      0.70448        0.930
 0.118     37       1   0.7875  0.0598      0.67860        0.914
 0.165     36       1   0.7656  0.0620      0.65324        0.897
 0.173     35       1   0.7437  0.0640      0.62835        0.880
 0.179     34       1   0.7219  0.0657      0.60388        0.863
 0.189     32       1   0.6993  0.0674      0.57888        0.845
 0.239     31       1   0.6768  0.0689      0.55428        0.826
 0.241     30       1   0.6542  0.0702      0.53007        0.807
```
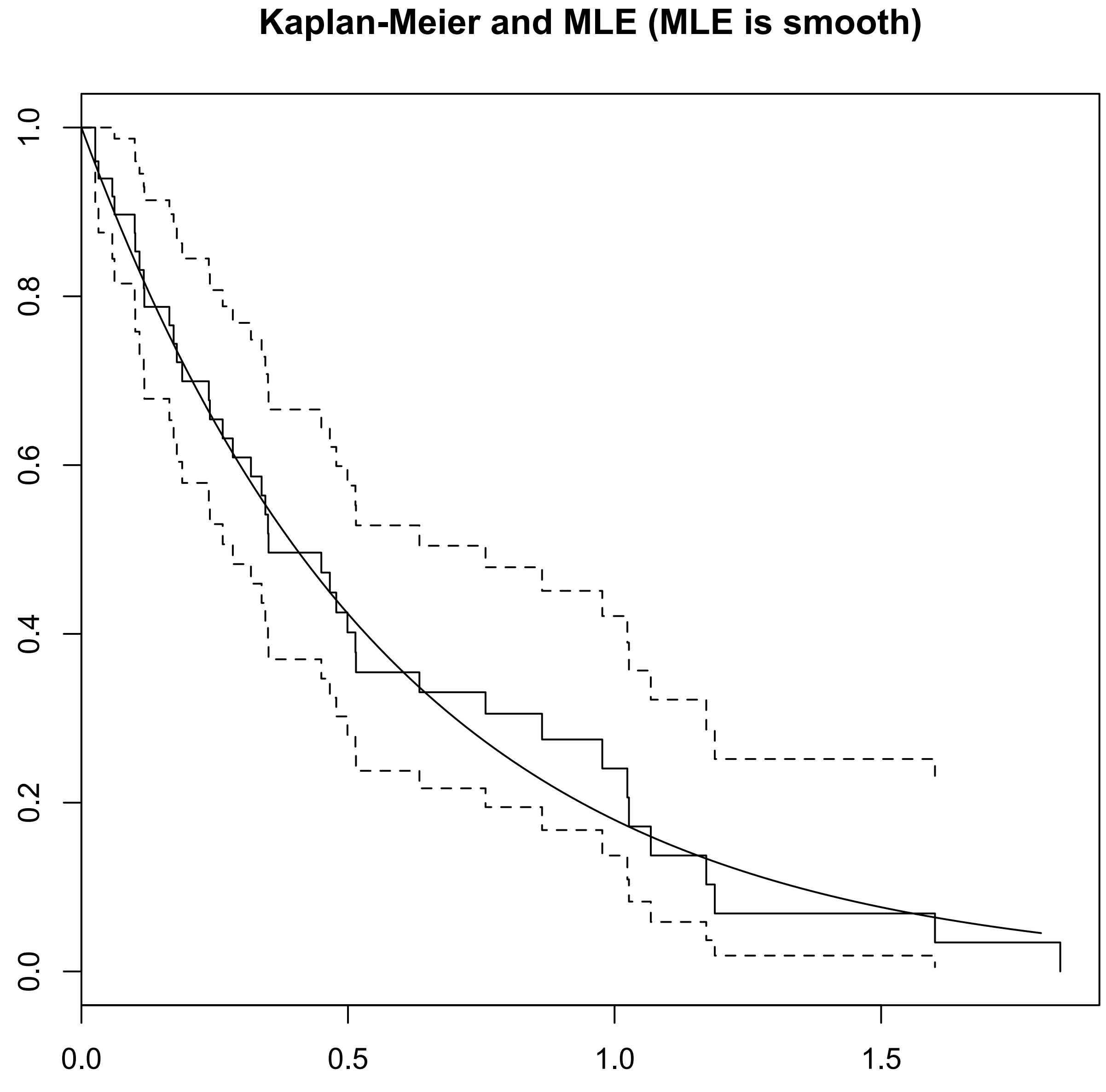

```
 0.265     29       1   0.6316  0.0713      0.50621        0.788
 0.284     28       1   0.6091  0.0723      0.48270        0.769
 0.318     27       1   0.5865  0.0730      0.45951        0.749
 0.338     26       1   0.5640  0.0736      0.43665        0.728
 0.345     25       1   0.5414  0.0741      0.41409        0.708
 0.350     24       1   0.5188  0.0743      0.39184        0.687
 0.351     23       1   0.4963  0.0744      0.36988        0.666
 0.450     21       1   0.4727  0.0745      0.34697        0.644
 0.466     20       1   0.4490  0.0745      0.32441        0.622
 0.478     19       1   0.4254  0.0742      0.30220        0.599
 0.499     18       1   0.4018  0.0738      0.28035        0.576
 0.514     17       1   0.3781  0.0731      0.25886        0.552
 0.515     16       1   0.3545  0.0723      0.23774        0.529
 0.634     15       1   0.3309  0.0712      0.21701        0.504
 0.758     13       1   0.3054  0.0701      0.19473        0.479
 0.864     10       1   0.2749  0.0694      0.16752        0.451
 0.977      8       1   0.2405  0.0687      0.13736        0.421
 1.024      7       1   0.2061  0.0670      0.10907        0.390
 1.027      6       1   0.1718  0.0640      0.08277        0.357
 1.068      5       1   0.1374  0.0597      0.05864        0.322
 1.172      4       1   0.1031  0.0538      0.03708        0.287
 1.188      3       1   0.0687  0.0455      0.01876        0.252
 1.601      2       1   0.0344  0.0333      0.00514        0.230
 1.836      1       1   0.0000     NaN           NA           NA
>
> # (b) S-hat(0:062) = 0.8969
>
> # (c) and (d)
> # p-hat     1       2       3       4
>          48/50 * 46/47 * 43/44 * 42/43
[1] 0.8968665
>
>
> # (e) This way of doing it requires you to realize sumkm is a list.
> #     The numbers could also be entered by hand.
> #     Try sumkm[1], sumkm[2] etc. to find out.
> n = sumkm$n.risk; d = sumkm$n.event; Shat = sumkm$surv
> Shat[4] # Another way to answer (b)
[1] 0.8968665
>
> se4 = Shat[4] * sqrt(sum(d[1:4]/(n[1:4]*(n[1:4]-d[1:4])))); se4 # 0.04373656
[1] 0.04373656
>
> # This should agree with the hand" calculation. Get 0.04373819
> 0.8969 * sqrt(sum(d[1:4]/(n[1:4]*(n[1:4]-d[1:4]))))
[1] 0.04373819
>
> # (f)
> plot(km)
>
>
> # 12
> # (a)
> # First, from last week (Assignment 5),
>
> # A5 Q2a) MLE
> lambdahat = sum(Uncensored)/sum(Time);  lambdahat
[1] 1.717107
>
> # A5 Q2b Estimated asymptotic variance
> vhat = lambdahat^2 / sum(Uncensored);  vhat # Estimated asymptotic variance
[1] 0.07371138
> se = sqrt(vhat); se
[1] 0.2714984
>
```

```
> # A5 Q2c) 95% CI for lambda: 0.95 =~ P(A < lambda < B)
> A = lambdahat - 1.96*se; B = lambdahat + 1.96*se
> c(A,B)
[1] 1.184970 2.249244
>
> # Now get to 12a from A6
> # Median of an exponential is log(2)/lambda
> medhat = log(2)/lambdahat; medhat
[1] 0.4036716
>
> # There are two ways to get a CI ...
>
> # Delta method
> semed = log(2)/(lambdahat*sqrt(sum(Uncensored))); semed
[1] 0.06382608
> lower95 = medhat - 1.96*semed; upper95 = medhat + 1.96*semed
> c(lower95,upper95)
[1] 0.2785725 0.5287707
>
> # Transform CI for lambda
> c(log(2)/B, log(2)/A)
[1] 0.3081690 0.5849492
>
>
>
> # (b)  Add MLE of S(t) to plot
> t = seq(from=0,to=1.8,length=101)
> Shat = exp(-lambdahat*t)
> lines(t,Shat)
> title('Kaplan-Meier and MLE (MLE is smooth)')
>
>
```


Kaplan-Meier and MLE (MLE is smooth)
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5