# 312f23 Assignment 3 (On the bus) 

(1) $$\ell(\mu, \sigma^2) = \log \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\log x_i - \mu)^2} \frac{1}{\log(x_i)}$$

$$= \log\left( (\sigma^2)^{-\frac{n}{2}} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)^2} \frac{1}{\prod_{i=1}^{n} \log x_i} \right)$$

$$= -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)^2 - \sum_{i=1}^{n} \log\log x_i$$

$$\frac{\partial \ell}{\partial \mu} \overset{0+0}{=} - \frac{1}{2\sigma^2}\sum_{i=1}^{n} \frac{\partial}{\partial\mu}(\log x_i - \mu)^2 - 0$$

$$= +\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n} \cancel{2}(\log x_i - \mu)(+1)$$

$$= \frac{1}{\sigma^2}\left(\sum_{i=1}^{n}\log x_i - n\mu\right) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n}\log x_i = n\mu \Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n}\log x_i}{n}$$

$$\frac{\partial \ell}{\partial \sigma^2} = \frac{\partial}{\partial\sigma^2}\left(-\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2}(\sigma^2)^{-1}\sum_{i=1}^{n}(\log x_i - \mu)^2 - \sum_{i=1}^{n}\log\log x_i\right)$$

$$= -\frac{n}{2\sigma^2} - \frac{1}{2}(-1)(\sigma^2)^{-2}\sum_{i=1}^{n}(\log x_i - \mu)^2 - 0$$

$$= -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^{n}(\log x_i - \mu)^2}{2\sigma^4} \overset{set}{=} 0$$

$$\Rightarrow \frac{n}{\cancel{2\sigma^2}} = \frac{\sum_{i=1}^{n}(\log x_i - \mu)^2}{\cancel{2\sigma^2}\sigma^2}$$

~~setting $\mu = \hat{\mu}$ & solving,~~

~~$(\hat{\mu}, \hat{\sigma}^2) = \left(\frac{\sum \log x_i}{n}, \frac{\sum_{i=1}^{n}(\log x_i - \hat{\mu})^2}{n}\right)$~~

(1a continued) Have

$$\mu = \frac{\sum_{i=1}^{n} \log t_i}{n}, \quad \sigma^2 = \frac{\sum_{i=1}^{n} (\log t_i - \mu)^2}{n}$$

Solving, have

$$(\hat{\mu}, \hat{\sigma}^2) = \left( \frac{\sum_{i=1}^{n} \log t_i}{n}, \frac{\sum_{i=1}^{n} (\log t_i - \mu)^2}{n} \right)$$

(b) with $n = 17$, mean(log(x)) = −1.263583
var(log(x)) = 5.114263,

$$\hat{\mu} = \text{mean}(\log(x)) = -1.263583$$

$$\hat{\sigma}^2 = \frac{16}{17} \text{var}(\log(x)) = 4.813424$$

2 (a) $\ell(\lambda) = \log \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \log\left(\frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\right)$

$= -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - \log \prod_{i=1}^{n} x_i!$

$\ell'(\lambda) = -n + \frac{\sum x_i}{\lambda} + 0 \overset{\text{set}}{=} 0$

$\Longrightarrow \frac{\sum_{i=1}^{n} x_i}{\lambda} = n \Longrightarrow \lambda = \frac{\sum x_i}{n} = \bar{x}$

Though in the problem, it's $\bar{y}$.

(b) $p = 6$

(c) $H_0: \lambda_1 = \lambda_2 = \cdots = \lambda_6$

(d) $\sum_{i=1}^{n} Y_i \sim \text{Poisson}(n\lambda)$

(e) $\text{Poisson}(n_j \lambda_j)$

(2f) $$L(\underset{\sim}{\lambda}) = \prod_{j=1}^{p} \prod_{i=1}^{n_j} \frac{e^{-\lambda_j} \lambda_j^{y_{ij}}}{y_{ij}!}$$

$$= \prod_{j=1}^{p} \frac{e^{-n_j \lambda_j} \lambda_j^{\sum_{i=1}^{n_j} y_{ij}}}{\prod_{i=1}^{n_j} y_{ij}!}$$

$$= \frac{e^{-\sum_{j=1}^{p} n_j \lambda_j} \prod_{j=1}^{p} \lambda_j^{\sum_{i=1}^{n_j} y_{ij}}}{\prod_{j=1}^{p} \prod_{i=1}^{n_j} y_{ij}!}$$

(g) $$\ell(\underset{\sim}{\lambda}) = -\sum_{j=1}^{p} n_j \lambda_j + \sum_{j=1}^{p} \left(\sum_{i=1}^{n_j} y_{ij}\right) \log \lambda_j - \sum_{j=1}^{p}\sum_{i=1}^{n_j} \log y_{ij}!$$

$$= -\sum_{j=1}^{p} n_j \lambda_j + \sum_{j=1}^{p} n_j \bar{y}_j \log \lambda_j - \sum_{j=1}^{p}\sum_{i=1}^{n_j} \log y_{ij}!$$

(h) $$\hat{\underset{\sim}{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_j) = (\bar{y}_1, \ldots, \bar{y}_j)$$

(2i) $\hat{\lambda}_0 = (\bar{y}, \bar{y}, \ldots, \bar{y})$

(j) $G^2 = -2 \log \dfrac{L(\hat{\lambda}_0)}{L(\hat{\lambda})}$

$$= -2 \log \frac{e^{-\sum_{j=1}^{p} n_j \bar{y}} \prod_{j=1}^{p} \bar{y}^{\,n_j \bar{y}_j} \Big/ \cancel{\prod \prod y_{ij}!}}{e^{-\sum_{j=1}^{p} n_j \bar{y}_j} \prod_{j=1}^{p} \bar{y}_j^{\,n_j \bar{y}_j} \Big/ \cancel{\prod \prod y_{ij}!}}$$

$$= -2 \log \frac{\cancel{e^{-\bar{y} \sum_{j=1}^{p} n_j}} \; \bar{y}^{\sum_{j=1}^{p} n_j \bar{y}_j}}{\cancel{e^{-\sum_{j=1}^{p} \sum_{i=1}^{n_j} y_{ij}}} \; \prod_{j=1}^{p} \bar{y}_j^{\,n_j \bar{y}_j}}$$

$$= -2 \left( \sum_{j=1}^{p} n_j \bar{y}_j \log \bar{y} - \sum_{j=1}^{p} n_j \bar{y}_j \log \bar{y}_j \right)$$

$$= 2 \left( \sum_{j=1}^{p} n_j \bar{y}_j \log \bar{y}_j - n \bar{y} \log \bar{y} \right)$$

(k) See printout

(l) <u>Yes</u>.

```
> # The 6 lines of code
> ybar = c(10.68,  9.87234,  9.56,  8.52, 10.48571,  9.98)
> n = c(50,        47,      50,     50,      35,        50)
> YBAR = sum(n*ybar)/sum(n); YBAR
[1] 9.815602
> N = sum(n); N
[1] 282
> G2 = 2 * ( sum(n*ybar*log(ybar)) - N*YBAR*log(YBAR) ); G2
[1] 14.7068
> pval = 1 - pchisq(G2,5); pval
[1] 0.01169142
>
```

③ (a) $\int_0^\infty \alpha \lambda (\lambda t)^{\alpha-1} \exp\{-(\lambda t)^\alpha\}\, dt$

$u = (\lambda t)^\alpha$ $du = \alpha (\lambda t)^{\alpha-1} \lambda \, dt$

| $t$ | $u$ |
|---|---|
| $\infty$ | $\infty$ |
| $0$ | $0$ |

$= \int_0^\infty e^{-u}\, du = 1$ (standard exponential)

(b) $\alpha = 1$

(c) $E(T^k) = \int_0^\infty t^k \exp\{-(\lambda t)^\alpha\} \alpha (\lambda t)^{\alpha-1} \lambda \, dt$

Again letting $u = (\lambda t)^\alpha = \lambda^\alpha t^\alpha \Rightarrow t^\alpha = \frac{u}{\lambda^\alpha}$

$\Rightarrow t = \left(\frac{u}{\lambda^\alpha}\right)^{\frac{1}{\alpha}} = \frac{u^{\frac{1}{\alpha}}}{\lambda}$, so

$E(T^k) = \int_0^\infty \left(\frac{u^{1/\alpha}}{\lambda}\right)^k e^{-u}\, du = \frac{1}{\lambda^k} \int_0^\infty e^{-u} u^{(\frac{k}{\alpha}+1)-1} du$

$= \frac{1}{\lambda^k} \cdot \frac{\Gamma(\frac{k}{\alpha}+1)}{1} \underbrace{\int_0^\infty \frac{1^{\frac{k}{\alpha}+1}}{\Gamma(\frac{k}{\alpha}+1)} e^{-u} u^{(\frac{k}{\alpha}+1)-1} du}_{=1}$

$= \frac{\Gamma(\frac{k}{\alpha}+1)}{\lambda^k}$

(3d) $F_T(x) = P(T \le x)$

$$= \int_0^x e^{-(\lambda t)^\alpha} \alpha (\lambda t)^{\alpha-1} \lambda \, dt$$

Again, let $u = (\lambda t)^\alpha$, $du = \alpha(\lambda t)^{\alpha-1}\lambda \, dt$

$$\begin{array}{c|c} t & u \\ \hline x & (\lambda x)^\alpha \\ 0 & 0 \end{array}$$

, so $F_T(x) = \int_0^{(\lambda x)^\alpha} e^{-u} \, du$

Using CDF of exponential

$$= 1 - e^{-(\lambda x)^\alpha} \stackrel{set}{=} \frac{1}{2}$$, solve for $x$

So $e^{-(\lambda x)^\alpha} = \frac{1}{2}$

$$\Rightarrow -\lambda^\alpha x^\alpha = \log \frac{1}{2} = -\log 2$$

$$\Rightarrow \lambda^\alpha x^\alpha = \log 2 \Rightarrow x^\alpha = \frac{\log 2}{\lambda^\alpha}$$

$$\Rightarrow x = \left(\frac{\log 2}{\lambda^\alpha}\right)^{1/\alpha} = \frac{(\log 2)^{1/\alpha}}{\lambda}$$

Median

To get CI for expected value: $\boxed{9}$

(3e iv) $E(T) = \dfrac{\Gamma(\frac{1}{\alpha}+1)}{\lambda} = g(\alpha,\lambda) = \Gamma(\alpha^{-1}+1)\,\lambda^{-1}$

$$\dot{g}(\alpha,\lambda) = \left(\frac{\partial g}{\partial \alpha}, \frac{\partial g}{\partial \lambda}\right)$$

$$= \left(\Gamma'(\tfrac{1}{\alpha}+1)(-1)\alpha^{-2}\frac{1}{\lambda},\ \Gamma(\tfrac{1}{\alpha}+1)(-1)\lambda^{-2}\right)$$

$$= \left(-\frac{\Gamma'(\frac{1}{\alpha}+1)}{\alpha^2\lambda},\ -\frac{\Gamma(\frac{1}{\alpha}+1)}{\lambda^2}\right)$$

(vi) To get CI for median $= \dfrac{(\log 2)^{\frac{1}{\alpha}}}{\lambda}$

$$\frac{\partial}{\partial\alpha}(\log 2)^{\frac{1}{\alpha}} = \frac{\partial}{\partial\alpha}\mathrm{Exp}\{\log(\log 2)^{\frac{1}{\alpha}}\}$$

$$= \frac{\partial}{\partial\alpha}\mathrm{Exp}\{\tfrac{1}{\alpha}\log(\log 2)\} = \frac{\partial}{\partial\alpha}\mathrm{Exp}\{\alpha^{-1}\log(\log 2)\}$$

$$= \mathrm{Exp}\{\tfrac{1}{\alpha}\log(\log 2)\}\cdot(-1)\alpha^{-2}\log(\log 2)$$

$$= \mathrm{Exp}\{\log(\log 2^{\frac{1}{\alpha}})\}\frac{-1}{\alpha^2}\log(\log 2)$$

$$= \frac{-(\log 2)^{\frac{1}{\alpha}}\log(\log 2)}{\alpha^2}, \text{ so}$$

$$\dot{g}_2(\alpha,\lambda) = \left(\frac{\partial g}{\partial\alpha}, \frac{\partial g}{\partial\lambda}\right) = \left(-\frac{(\log 2)^{\frac{1}{\alpha}}\log(\log 2)}{\lambda\alpha^2},\ -\frac{(\log 2)^{1/\alpha}}{\lambda^2}\right)$$

# STA312f23 Assignment 3 Q3e Printout

```
> # Assignment 3, Question 3e.
> rm(list=ls()); options(scipen=999)
> x = scan("http://www.utstat.toronto.edu/brunner/data/legal/Weibull.data1.txt")
Read 500 items
>
> # (i) Find MLE
> mloglike = function(theta,datta)
+     {
+     alpha = theta[1]; lambda = theta[2]
+     n = length(datta)
+     value = lambda^alpha*sum(datta^alpha) -
+             n*log(alpha) - n*alpha*log(lambda) - (alpha-1)*sum(log(datta))
+     return(value)
+     } # End of function mloglike
>
> # Testing
> mloglike(c(2,0.25),datta=x) # 1019.647
[1] 1019.647
> -sum(dweibull(x,shape=2,scale = 4,log=TRUE)) # 1019.647
[1] 1019.647
>
> # Find MLE: Truth is alpha = 2 and lambda=1/4
> startvals = c(1,1)
> search1 = optim(par=startvals, fn=mloglike, datta=x,
+                 hessian=TRUE, lower=c(0,0), method='L-BFGS-B')
> search1
$par
[1] 1.9124466 0.2454345

$value
[1] 1018.006

$counts
function gradient
      13       13

$convergence
[1] 0

$message
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"

$hessian
         [,1]       [,2]
[1,] 244.4778   843.0713
[2,] 843.0713 30358.9099

>
> alphahat = search1$par[1]; alphahat # Truth is 2
[1] 1.912447
> lambdahat = search1$par[2]; lambdahat # Truth is 1/4
[1] 0.2454345
> Vhat = solve(search1$hessian); Vhat
              [,1]           [,2]
[1,]  0.0045235447 -0.00012561949
[2,] -0.0001256195  0.00003642773
>
```

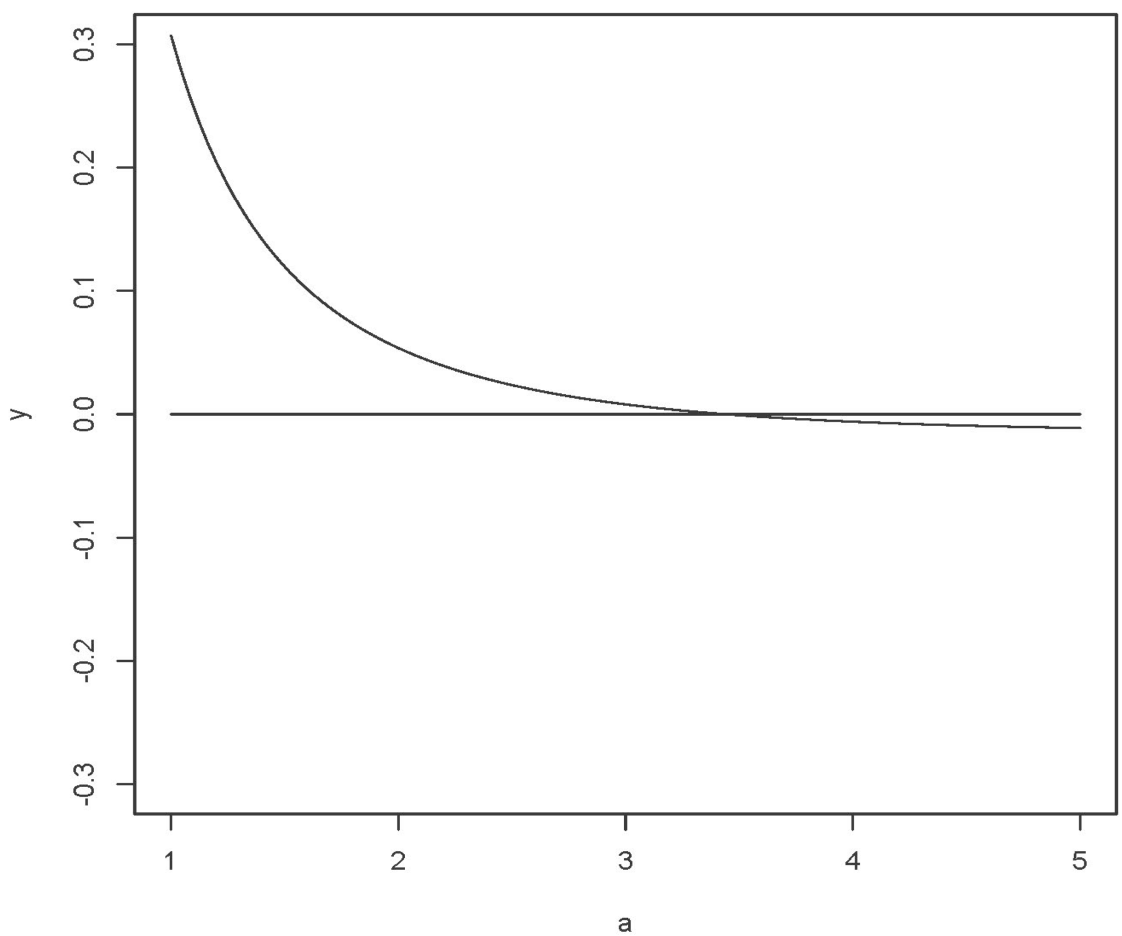
```
>
> # (ii) 95% CI for alpha
>
> # CI for alpha
> se_alphahat = sqrt(Vhat[1,1])
> lower95 = alphahat - 1.96*se_alphahat; upper95 = alphahat + 1.96*se_alphahat
> c(lower95,upper95)
[1] 1.780622 2.044271
>
> # (iii) Point estimate of expected value
>
> # Give a point estimate of the expected value (mu = gamma(1+1/alpha)/lambda).
Truth is 3.544908
> muhat = gamma(1+1/alphahat)/lambdahat; muhat
[1] 3.614746
> mean(x) # Compare
[1] 3.618727
>
> # (iv) 95% CI for mu
> # Hint is help(digamma) for gdot
>
> gprime = digamma(1+1/alphahat)*gamma(1+1/alphahat)
> gdot = cbind( -gprime/(alphahat^2*lambdahat), -gamma(1+1/alphahat)/lambdahat^2)
> v_muhat = as.numeric( gdot %*% Vhat %*% t(gdot) ); se_muhat = sqrt(v_muhat)
> lower95 = muhat - 1.96*se_muhat; upper95 = muhat + 1.96*se_muhat
> c(lower95,muhat,upper95)
[1] 3.442696 3.614746 3.786796
>
> t.test(x) # For comparison

	One Sample t-test

data:  x
t = 41.284, df = 499, p-value < 0.00000000000000022
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 3.446509 3.790946
sample estimates:
mean of x
 3.618727

>
> # (v) Estimate median
>
> mhat = 1/lambdahat * log(2)^(1/alphahat); mhat # Truth is 3.544908
[1] 3.363827
> median(x) # Sample median
[1] 3.411699
>
> # (vi) 95% CI for median
> # D[b^(1/a),a] works in Wolfram Alpha, hand as a check.
>
> gdot2 = cbind( - log(2)^(1/alphahat)*log(log(2))/(lambdahat*alphahat^2),
+                - log(2)^(1/alphahat)/lambdahat^2 )
> v_mhat = as.numeric( gdot2 %*% Vhat %*% t(gdot2) ); se_mhat = sqrt(v_mhat)
> lower95 = mhat - 1.96*se_mhat; upper95 = mhat + 1.96*se_mhat
> c(lower95,upper95)
[1] 3.182938 3.544715
>
> # > ci.median(x) # From asbio package
> # 95% Confidence interval for population median
> # Estimate     2.5%    97.5%
> # 3.411699 3.176731 3.624114
>
```

```
>
> # (vii) Test mean = median
>
> # Straightforward delta method.
> gdot3 = gdot-gdot2 # Derivaltive of a difference is difference of derivatives.
> v_diff1 = as.numeric( gdot3 %*% Vhat %*% t(gdot3) ); se_diff1 = sqrt(v_diff1)
>
> Z1 = (muhat-mhat)/se_diff1;  Z1
[1] 9.923228
>
> # The other methods simplify the null hypothesis, and test a simple
> # equivalent statement.
> # I believe gamma(1+1/alpha) = log2^(1/alpha) iff alpha has a particular
> # numerical value,
> # the root of g(alpha) = gamma(1+1/alpha) - log2^(1/alpha)
>
> # Plot it to see if it has a root.
> a = seq(from=1,to=5,length=101)
> y = gamma(1+1/a) - log(2)^(1/a)
> plot(a,y,type='l', ylim=c(-0.3,.3))
> xx = c(1,5); yy = c(0,0); lines(xx,yy)
> # Root is between alpha=3 and alpha=4
```



```
>
> g = function(x) gamma(1+1/x) - log(2)^(1/x)
> intersection = uniroot(g,c(3,4)); intersection
$root
[1] 3.439545

$f.root
[1] -0.00000006785751

$iter
[1] 5

$init.it
[1] NA

$estim.prec
[1] 0.00006103516

> # Already see reject H0 because it's outside 95% CI for alpha
> alpha0 = intersection$root; alpha0
[1] 3.439545
>
> Z2 = (alphahat-alpha0)/se_alphahat; Z2
[1] -22.70532
>
> # LR test of H0: alpha=alpha0. Need restricted MLE lambdahat0
>
> restmll = function(lambda,datta) # Restricted minus loglike
+     {
+     alpha = alpha0
+     n = length(datta)
+     value = lambda^alpha*sum(datta^alpha) -
+             n*log(alpha) - n*alpha*log(lambda) - (alpha-1)*sum(log(datta))
+     if(value>10^10) value = 10^10
+     return(value)
+     } # End of function restmll
>
> # Try some values
>
> restmll(.25,datta=x)
[1] 1305.871
> restmll(1,datta=x)
[1] 101511.3
> restmll(.001,datta=x)
[1] 9922.515
> # That brackets it
>
> search2 = optim(par=lambdahat, fn=restmll, datta=x,
+                 lower=0, upper=1, method='L-BFGS-B')
> search2
$par
[1] 0.2121786

$value
[1] 1208.953

$counts
function gradient
       6        6

$convergence
[1] 0

$message
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"
```

```
>
> # G-squared is twice difference between minus log likelihoods
> # And of course the minus LL (lack of fit) is greater for the restricted model.
>
> Gsq = 2 * (search2$value - search1$value); Gsq
[1] 381.8942
>
> round(c(Z1,Z2,Gsq),2)
[1]   9.92 -22.71 381.89
> Z2^2 # The Wald statistc for H0: alpha=alpha0
[1] 515.5317
```