STA 312f23 Assignment Two[1]

The paper and pencil questions are not to be handed in. They are practice for Quiz 2 on September 22nd. The R part of Question 9 may be handed in as part of the quiz. **Bring your printout to the quiz**. Do not write anything on your printout in advance except possibly your name and student number.

1. This question is a repeat from Assignment One. If you have already done it, great!

   For each of the following distributions, derive a general expression for the Maximum Likelihood Estimator (MLE). Carry out the second derivative test to make sure you really have a maximum. Then use the data to calculate a numerical estimate.

   (a) $p(x) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$. Data: `4, 0, 1, 0, 1, 3, 2, 16, 3, 0, 4, 3, 6, 16, 0, 0, 1, 1, 6, 10`. Answer: 0.2061856

   (b) $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$, where $\alpha > 0$. Data: `1.37, 2.89, 1.52, 1.77, 1.04, 2.71, 1.19, 1.13, 15.66, 1.43`. Answer: 1.469102

   (c) $f(x) = \frac{\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2 x^2}{2}}$, for x real, where $\tau > 0$. Data: `1.45, 0.47, -3.33, 0.82, -1.59, -0.37, -1.56, -0.20` . Answer: 0.6451059

   (d) $f(x) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. Data: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778

2. For Problem 1c,

   (a) Use your work on the second derivative teat to produce a formula for the estimated asymptotic variance of the MLE. You will have the posted formula sheet for Quiz 2.

   (b) Give a numerical standard error for the MLE.

3. Let X have a Bernoulli distribution with parameter θ. Verify the formulas for expected value and variance on the formula sheet.

4. Let X have an exponential distribution with parameter λ. Verify the formulas for expected value and variance on the formula sheet. To save some work, you may use the fact that all the densities on the formula sheet integrate to one.

5. Let $X \sim N(\mu, \sigma^2)$. Show $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.

6. Prove that if $Z \sim N(0,1)$, then $Z^2 \sim \chi^2(1)$. You may use the fact that a $\chi^2(\nu)$ random variable is gamma, with $\alpha = \frac{\nu}{2}$ and $\beta = \frac{1}{2}$.

7. Let $X_1, \ldots, X_n$ be a random sample (that is, independent and identically distributed) from a distribution with expected value μ and variance σ^2. The sample mean is $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) Calculate $E(\overline{X}_n)$. Show your work.

   (b) Calculate $Var(\overline{X}_n)$. Show your work.

8. Let $X_1, \ldots, X_n$ be a random sample (that is, independent and identically distributed) from a Poisson distribution with parameter $\lambda > 0$. The sample mean for a sample of $n = 49$ is $\overline{X} = 4.2$.

   (a) Derive a formula for $\widehat{\lambda}$, the maximum likelihood estimate of λ.

   (b) Carry out the second derivative test.

   (c) Give a point estimate of λ. Your answer is a number.

   (d) Give a 95% confidence interval for λ; the answer is a pair of numbers. My lower confidence limit is 3.63. Note that in this problem you do *not* need to go through the Fisher information to get the standard error, because you can just write down the exact variance of $\widehat{\lambda}$.

   (e) Carry out a two-sided Z-test of $H_0 : \lambda = 3$ at $\alpha = 0.05$.

      i. What is the critical value? The answer is a number You can use R, or just look it up on the lecture slides.

      ii. Calculate the test statistic. The answer is a number. I did it two ways and got $Z = 4.1$ and $Z = 4.85$. The method yielding 4.1 is more similar to what we will be doing later in the course.

      iii. Do you reject the null hypothesis? Answer Yes or No.

      iv. Do you conclude that λ is different from 3? Answer Yes or No.

      v. If the answer to the last question was Yes, do you conclude that λ is less than 3, or that λ is greater than 3? Pick one.

      vi. Use R to calculate the two-sided p-value for $Z = 4.1$. My answer is 4.131501e-05 = 0.00004131501.

9. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\pi) = \pi e^{-\pi/x} \frac{1}{x^2}$ for $x > 0$, and zero for $x \leq 0$. The unknown parameter π is greater than zero.

   (a) Verify that this really is a density by showing that it integrates to one. To save some work, you may use the fact that all the densities on the formula sheet integrate to one.

   (b) Derive a formula for the MLE of π. Include the second derivative test. Show your work and circle your final answer.

   (c) Give a formula for $\widehat{v}_n$, the estimated asymptotic variance of $\widehat{\pi}_n$. Show a little work.

   (d) The file `http://www.utstat.toronto.edu/brunner/data/legal/inversegamma.data.txt` has a set of raw data. Using R and your answers to Questions 9b and 9c, calculate

      i. The maximum likelihood estimate $\widehat{\pi}_n$.
      ii. A 95% confidence interval for π.

      The answers are numbers on your printout.

   (e) Test $H_0 : \pi = 3.14159$ with a two-sided large-sample Z-test, using the $\alpha = 0.05$ significance level.

      i. There are two critical values, one for the lower tail and one for the upper tail. What are they? The answers are numbers.
      ii. What is the value of the test statistic? The answer is a number on your printout.
      iii. Use R to calculate the 2-sided p-value. The answer is a number on your printout. My answer is `0.01125527`.
      iv. Do you reject the null hypothesis? Answer Yes or No.
      v. Are the results statistically significant? Answer Yes or No.
      vi. Do these data contradict claim that $\pi = 3.14159$? Answer Yes or No.

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