Randomness and the Riemann Hypothesis

Probability, Statistics, and the Primes

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Preface

Only once before did I stand in a comparable awe before the mysterious harmony of thought and nature. That was when together with Erdős in the days of our youth we had found that primes play a game of chance.

Marek (Mark) Kac (1979, [**614**])

We have written this book with three closely related aims in mind.

First and foremost, we wanted to retell one of the most remarkable mathematical stories ever told. But we wanted to tell it through what has proven, time and again, to be a singularly fruitful lens, one characterized by notions of randomness and by probabilistic and statistical reasoning. This is a viewpoint that has led to enormous progress with spectacular successes along the way, as well as occasional failures which have in equal parts been as surprising as they are enlightening. It is a viewpoint that has served as an ofttimes reluctant backbone for the subject over hundreds of years and yet has guided and challenged the intuitions of what have unquestionably been some of the most able explorers who ever lived.

Secondly, we wanted to tell this story in a radically different way, and for it to be as self-contained and as accessible as possible to a broad class of readers who have some mathematical background. In particular, our hope is that such readers will find it more rewarding to follow a treatment such as this rather than one intended for a lay audience. But at the same time, we wanted to tell the story without pulling any of its punches; to tell it in a way that is authentic and, in certain respects, complete. Nevertheless, this book has not been written for experts (much as we would be gratified if some were to find it useful) and neither was it meant to be a textbook. As *first* envisioned, our expectation of the reader was to have just a little more than the maturity gained from a *rigorous* course in calculus—one that included the rudiments of real analysis—together with an adventurous spirit and a desire to expand horizons. However, as our project grew it became evident that more than this—in particular, having encountered some rigorous probability as well as some complex analysis—would be desirable as well. But since—initially, at least—we had never meant to *demand* this of the reader, we have made amends. These appear in the form of self-contained and carefully devised appendices that cover these topics (among others) in lively and meticulous detail. The appendices offer opportunities for extending and solidifying one's understanding of the material they cover, and constitute an integral part of this book. To be sure, readers with less background will encounter passages they may find challenging. But we encourage them to acknowledge and set such difficulties aside, and skip to other sections and

other chapters. Such passages can be returned to over time after more background and experience have been acquired.

Our third aim was to pay homage to the central construct that has driven this subject for nearly two hundred years, namely the Riemann zeta function, and the seemingly impenetrable mysteries concerning its zeros. Therefore, as much as our perspective was meant to centre on notions of randomness and probability, we have nevertheless paid considerable attention to the zeta function throughout. Needless to say, however, the probabilistic aspects of this function have not been ignored.

This book is the result of what was at first meant to be a much smaller endeavour, a labour of love attempting to capture the grandeur of its subject matter. For one author this project was undertaken as a way to learn this remarkable subject and to faithfully record that journey; for the other it was a way to share some beautiful things acquired over many years (whilst holding the other author in check). But as this project progressed it grew, and it became clear to us that it could serve a broader purpose than originally intended. We realized that it could, on the one hand, become a work that a mathematically seasoned reader would enjoy reading over an extended period of time, purely for pleasure. But we realized too that it might serve as a flexible and uninhibiting resource for certain kinds of courses, one that would allow an adventurous instructor to plot their own flight path, augmenting the material with some of their own favourite topics, while offering students what we hope is a book they will *want* to read.

But there is more. Our book has turned out to include a considerable amount of detail, with many of its topics covered rather intensively. As a consequence, this book can also serve as a handy reference for a broad range of material, although more casual readers are, of course, free to decide which topics they wish to pursue in depth, and which to read more lightly. Furthermore, in our efforts to keep this book as self-contained as possible, we have included the previously mentioned carefully prepared appendices, most of which can—and should—be read attentively. These can be used for study, for review, for envisioning the big pictures, or to be savoured solely for the pleasure of their content. At the same time, the appendices also include useful material that, in many cases, is widely strewn across the literature. More about what this book contains and how it is organized is spelled out in Chapter 1; in particular, Section 1.2 provides guidance on the best ways to navigate the material.

We have avoided the temptation to compete with any of the splendid accounts that are now available of number theory, and of analytic or probabilistic number theory in particular. We have therefore refrained from attempting to reprove every last result from among those widely available elsewhere, although references for proofs that have been omitted are liberally provided. Nevertheless, for readers who are game, we have retained a judicious selection of proofs, particularly those that highlight the statistical and probabilistic aspects of the subject. Otherwise, we have aimed to be resolutely rigorous (but without being unduly pedantic) and to quote best, or close-to-best, results whenever possible, mindful of the value this can have when serving as a reference. We have selected our coverage to stay as close as possible to our main themes, namely probabilistic and statistical thinking about the primes, about the Riemann zeta function, and about other closely related aspects of the theory. Above all, we did aim "to write an interesting book, and one unlike other books", one that will be worthwhile as well as fun to read [**499**, Preface]. And

if we may further quote Hardy & Wright—with no presumption intended!—"We may have succeeded at the price of too much eccentricity, or we may have failed; but we can hardly have failed completely, the subject matter being so attractive that only extravagant incompetence could make it dull" [**499**, Preface]. Needless to say, however, while we did intend for our scope to be broad, it could not be unlimited, and not every important topic could be covered.

And now, we invite the reader: *read on, and enjoy*!