Lecture 2

Probability and Counting Rules

Sample-Point Method:

1. Define the experiment and describe a sample space, *S*.
2. List all the simple events.
3. Assign probabilities to the sample points in *S*.
   $P(E_i) \geq 0$ and $\sum P(E_i) = 1$
4. Define the event *A* as a collection of sample points.
5. Calculate *P*(*A*) by summing the probabilities of sample points in *A*.

Example: Toss a coin 3 times. Find *P*(of exactly 2 heads).

Solution: 1. Observe outcome of 3 tosses

2. $E_1 = HHH$, $E_2 = HHT$, $E_3 = THT$
   $E_4 = THH$, $E_5 = HTT$, $E_6 = HTH$.
   $E_7 = TTH$, $E_8 = TTT$

3. $P(E_i) = \frac{1}{8}$

4. $A = \{2 \text{ heads}\} = \{E_2, E_4, E_6\}$

5. $P(A) = P(E_2) + P(E_4) + P(E_6)$
   $= 3/8$

How to count sample points?

<u>Theorem</u> (*mn*-rule): With m elements $a_1, a_2, \dots, a_m$ and n elements $b_1, b_2, \dots, b_m$, it is possible to form $mn = m \times n$ pairs containing one element from each group.

<u>Proof</u>:

follows from $\longrightarrow$

	a_1	a_2	$\dots$	a_m
b_1	(a_1, b_1)			
b_2				
$\vdots$				
b_n				

Given $a_1, \dots, a_m$; $b_1, \dots, b_n$; $c_1, \dots, c_p$

$\Rightarrow$ # of triplets is mnp.

<u>Example</u>: Toss a coin 3 times.

a_1	a_2	b_1	b_2	c_1	c_2
T	H	T	H	T	H

# of triplets $= 2 \cdot 2 \cdot 2 = 2^3 = 8$

<u>Birthday Example</u>: We record the birthdays for each of 20 randomly selected persons. Assuming there are 365 possible distinct birthdays, find the number of points in the sample space S for this experiment. What is *P*(each person has a different birthday)?

<u>Solution</u>: Card(A) = |A| = # of elements in A

cardinality

$A = \{1, 2, 3, 4\}$, $|A| = 4$

Card(S) = # of 20-tuples

$$= \underbrace{365 \cdot 365 \cdot \ldots \cdot 365}_{20 \text{ times}} = 365^{20},$$

$$S = \bigcup_{i=1}^{365^{20}} E_i, \quad P(E_i) = 1/(365)^{20}$$

A = {each person has a diff. birthday}

$$P(A) = \frac{\text{Card}(A)}{\text{Card}(S)} = \frac{?}{365^{20}}$$

$$\text{Card}(A) = 365 \cdot 364 \cdot 363 \cdot \ldots \cdot 346$$

$$P(A) = \frac{365 \cdot 364 \cdot 363 \cdot \ldots \cdot 346}{365^{20}}$$

$$= 0.5886$$

Definition: An ordered arrangement of distinct objects is called a **permutation**.

Denote P_r^n = number of ways of ordering *n* distinct objects taken *r* at a time.

Theorem: $P_r^n = n(n-1)(n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$

Proof: $n(n-1)(n-2) \cdot \ldots \cdot (n-r+1) \cdot \frac{(n-r)!}{(n-r)!}$

$$= \frac{n!}{(n-r)!}$$

$$n! = n(n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

$$0! = 1$$

Definition: The number of **combinations** of *n* objects taken *r* at a time is the number of subsets, each of size *r*, that can be formed from n objects.

Denote $C_r^n = \binom{n}{r}$ = number of combinations.

Theorem: $C_r^n = \binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$

Example: Two cards are drawn from a 52-card deck. What is P(ace and face card)?

Solution: $\text{card}(S) = \binom{52}{2} = 1326$

$A = \{\text{an ace and a face card}\}$

$\text{card}\, A = \binom{4}{1}\binom{12}{1} = 48$

$P(A) = \frac{48}{1326} = 0.0362$

Example (#2.64): Toss a die 6 times. Find the probability of observing 1, 2, 3, 4, 5, and 6 in any order. $= A$

Solution: $(1\ 1\ 2\ 3\ 1\ 4),\ (2\ 4\ 6\ 3\ 5\ 1)$

. . . .

$P_6^6 = 6!$ $\quad P(A) = 6! \cdot \left(\frac{1}{6}\right)^6 = \frac{5}{324}$

Conditional Probability and Independence of Events

Definition: The **conditional probability** of an event A, given that an event B has occurred is given by

$$P(A|B) = \frac{P(A\cap B)}{P(B)},\ P(B)>0$$

Example: We toss a die once. Find a probability of a 1, given that an odd number was obtained.

Solution:

$A = \{\text{observe '1'}\} = \{1\}$

$B = \{\text{observe odd number}\} = \{1,3,5\}$

$A\cap B = \{1\} = A \Rightarrow P(A\cap B) = P(A) = \frac{1}{6}$

$P(B) = \frac{1}{2}$

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$

Definition: Two events A and B are said to be **independent** if any one of the following holds:

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

$$P(A \cap B) = P(A)P(B)$$

$$P(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A)P(B)$$

Example: Toss a die. Let A={observe an odd number}, B={observe an even number}, C={observe 1 or 2}.

$$A = \{1, 3, 5\}; \quad B = \{2, 4, 6\}, \quad C = \{1, 2\}$$

$$P(A) = 1/2 = P(B), \quad A \cap B = \emptyset$$

$$\Rightarrow P(A \cap B) = 0 \neq P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$\Rightarrow$ A, B are dependent

$$P(C) = \frac{1}{3}, \quad A \cap C = \{1\} \Rightarrow P(A \cap C) = \frac{1}{6}$$

$$P(A \cap C) = P(A)P(C) = \frac{1}{6}$$

$\Rightarrow$ A, C are independent

Note: ‘Mutually exclusive’ ≠ ‘independent’.

Two Laws of Probability

Theorem (The Multiplicative Law):

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

If A, B are independent,

$$P(A \cap B) = P(A)P(B)$$

Proof: follows from the definition

$$P(A \cap B \cap C) = P((A \cap B) \cap C) = P(A \cap B)P(C | A \cap B)$$

$$= P(A)P(B|A)P(C | A \cap B)$$

In general,

$$P(A_1 \cap A_2 \cap \dots \cap A_k)$$

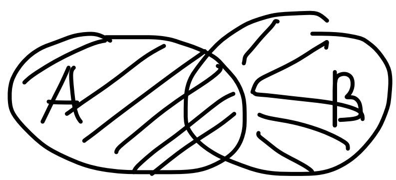
$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P\left(A_k \middle| \bigcap_{i=1}^{k-1} A_i\right)$$

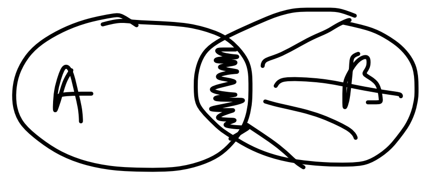
Theorem (The Additive Law):

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

If $A\cap B = \emptyset \Rightarrow P(A\cup B) = P(A) + P(B)$

Proof:





$$A\cup B = A\cup(B\cap\bar{A}), \quad A\cap(B\cap\bar{A}) = \emptyset$$

$$B = (A\cap B)\cup(B\cap\bar{A})$$

$$(A\cap B)\cap(B\cap\bar{A}) = \emptyset$$

$$P(A\cup B) = P(A) + P(B\cap\bar{A})$$

$$-\quad P(B) = P(A\cap B) + P(B\cap\bar{A})$$

$$P(A\cup B) - P(B) = P(A) - P(A\cap B)$$

$$\Rightarrow P(A\cup B) = P(A) + P(B) - P(A\cap B) \quad \blacksquare$$

Theorem $P(A) = 1 - P(\bar{A})$

Pf: $A\cup\bar{A} = S \Rightarrow P(A\cup\bar{A}) = P(A) + P(\bar{A}) = 1$

$A\cap\bar{A} = \emptyset$

$\Rightarrow P(A) = 1 - P(\bar{A}) \quad \blacksquare$

The Event Composition Method:

1. Define the experiment.
2. Describe the sample space.
3. Write the equation that expresses the event A as a composition of two or more events.
4. Apply the additive and multiplicative laws of probability.

Example: A patient with a disease will respond to treatment with probability of 0.9. If three patients are treated and respond independently, find P(at least one will respond).

Solution: $A = \{\text{at least one}\}$

$B_1 = \{1^{st}$ patient will not respond$\}$

$B_2 = \{2^{nd} \ldots \}$

$B_3 = \{3^{rd} \ldots \}$

$$\bar{A} = B_1 \cap B_2 \cap B_3$$

$$P(A) = 1 - P(B_1 \cap B_2 \cap B_3)$$

$$= 1 - P(B_1)P(B_2 | B_1)P(B_3 | B_1 \cap B_2)$$

$$= 1 - P(B_1)P(B_2)P(B_3) = 1 - 0.1^3$$

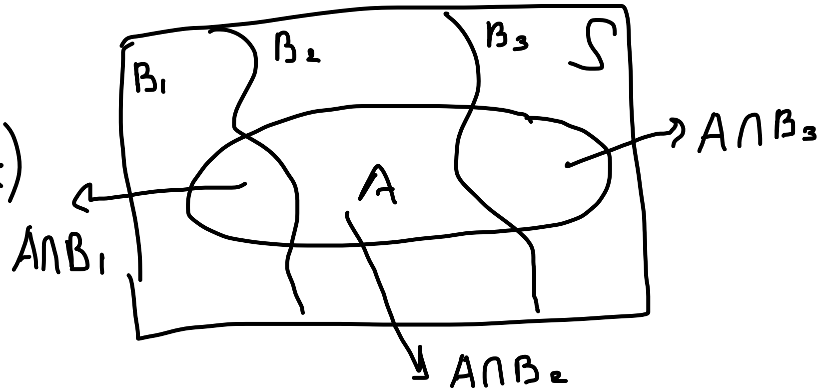
$$= 0.999$$

The Law of Total Probability and Bayes' Rule

Definition: For some $k \in Z^+$, let the sets $B_1, B_2, \dots, B_k$ be such that $S = B_1 \cup B_2 \cup \dots \cup B_k$, $B_i \cap B_j = \emptyset$, $i \neq j$. Then the collection of sets $\{B_1, B_2, \dots, B_k\}$ is said to be a **partition** of S.

Any set $A \subset S$

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_k)$$



Theorem: Assume that $\{B_1, B_2, \dots, B_k\}$ is a partition of S such that $P(B_i) > 0$, $i = 1, \dots, k$. Then for any event A,

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i).$$

Proof:

$$A = (A \cap B_1) \cup \dots \cup (A \cap B_k)$$

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset, \quad i \neq j$$

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i)$$

$$= \sum_{i=1}^{k} P(A|B_i)P(B_i)$$

Theorem (Bayes' Rule): Assume $\{B_1, B_2, \dots, B_k\}$ is a partition of S such that $P(B_i) > 0,\ i = 1, \dots, k$. Then

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}.$$

Proof:

$$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}$$

Example: (#2.136) A personnel director has two lists of applicants for jobs. List 1 contains the names of 5 women and 2 men, list 2 contains the names of 2 women and 6 men. A name is randomly selected from list 1 and added to list 2. A name is then randomly selected from the augmented list 2. Given that the name selected is that of a man, what is the probability that a woman's name was originally selected from list 1?

Solution:

list 1: 5w 2m

list 2: 2w 6m

A = {man's name selected from list 2

B = {woman's name selected from list 1

$P(B|A) = ?$

$B, \bar{B}$ - partition of S

$B_1 \quad B_2$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

$$P(B) = \frac{5}{7}, \quad P(A|B) = \frac{2}{3}$$

$$P(A|\bar{B}) = \frac{7}{9}, \quad P(\bar{B}) = \frac{2}{7}$$

$$P(B|A) = \frac{2/3 \cdot 5/7}{2/3 \cdot 5/7 + \frac{7}{9} \cdot \frac{2}{7}} = \frac{\frac{10}{21}}{\frac{10}{21} + \frac{14}{63}} = \frac{15}{22}$$

Random Variables

Definition: A **random variable** (r.v.) is a real-valued function for which the domain is a sample space.

Example: We toss 2 coins. Let *Y* equal the number of heads.

$$S = \{\underset{E_1}{HH}, \underset{E_2}{HT}, \underset{E_3}{TH}, \underset{E_4}{TT}\}$$

$$\{Y=0\} = \{E_4\}, \quad \{Y=1\} = \{E_2, E_3\}$$

$$\{Y=2\} = \{E_1\}$$

Let *y* denote an observed value of *Y*. Then *P*(*Y*=*y*) is the sum of the probabilities of the sample points that are assigned the value *y*.

$$P(Y=0) = P(E_4) = \frac{1}{4}$$

$$P(Y=1) = P(E_2) + P(E_3) = \frac{1}{2}$$

$$P(Y=2) = P(E_1) = 1/4$$