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Student Number

STA 442/2101 f2014 Quiz 6

1. For a simple linear regression with an intercept and one explanatory variable, you obtain the least squares estimates by minimizing $Q(\beta_0, \beta_1) = \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2$.

   (a) (2 points) Differentiate Q with respect to β_0 and set the derivative to zero and simplify a bit, obtaining the first *normal equation.*

$$\frac{\partial Q}{\partial \beta_0} = \sum_{i=1}^{n} 2(Y_i - \beta_0 - \beta_1 x_i)(-1) \stackrel{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} Y_i = n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i$$

   (b) (3 points) Noting that the quantities $\widehat{\beta}_0$ and $\widehat{\beta}_1$ must satisfy the first normal equation, show that the least squares line passes through the point $(\overline{x}, \overline{Y})$. *Start by giving the equation of the least squares line* in terms of $\widehat{\beta}_0$ and $\widehat{\beta}_1$.

Equation of the least squares line is $y = \hat{\beta}_0 + \hat{\beta}_1 x$.

From the first normal equation,

$$\sum_{i=1}^{n} Y_i = n\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{n} x_i \Rightarrow \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{n}{n}\hat{\beta}_0 + \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n}$$

$$\Rightarrow \overline{Y} = \hat{\beta}_0 + \hat{\beta}_1 \overline{x}$$ and the line goes through $(\overline{x}, \overline{Y})$

2. (2 points) The density of the t distribution is symmetric around zero, so that if $T \sim t(\nu)$, then also $-T \sim t(\nu)$. This fact makes it possible for a normal person to just write down confidence intervals and prediction intervals from the formula sheet without showing any work. Based on a regression with n cases, give a $(1-\alpha)100\%$ prediction interval for Y_{n+1}. Use $t_{\alpha/2}$ for the point satisfying $P\{T > t_{\alpha/2}\} = \alpha/2$. *You don't have to show any work.* Give it in the form $1-\alpha$ equals the probability of something.

$$1-\alpha = Pr\Big\{ x_{n+1}^T \hat{\beta} - t_{\alpha/2}\sqrt{MSE\,(1+x_{n+1}^T (X^TX)^{-1} x_{n+1}^T)} < Y_{n+1} < x_{n+1}^T \hat{\beta} + t_{\alpha/2}\sqrt{MSE\,(1+x_{n+1}^T (X^TX)^{-1} x_{n+1})} \Big\}$$

3. (3 points) Of course the explanatory variable values are often random, and *not* fixed constants. Suppose the explanatory variables are random. Is the prediction interval still valid? **Answer Yes or No** and show your work. For convenience of notation, you may pretend that the joint distribution of the **X** matrix and x_{n+1} is continuous.

To avoid a lot of writing, let $A(W)$ denote the lower limit of your prediction interval, and let $B(W)$ denote the upper limit.

$$Pr\{A(W) < Y_{n+1} < B(W)\}$$

$$= \int \cdots \int P\{A(W) < Y_{n+1} < B(W) \,|\, W = w\} f(w)\, dw$$

$$= \int \cdots \int (1-\alpha) f(w)\, dw = (1-\alpha) \int \cdots \int f(w)\, dw$$

$$= (1-\alpha) \cdot 1 = 1-\alpha$$

<u>YES</u>