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Student Number

# STA 442/2101 f2014 Quiz 2

1. (5 points) Independently for $i = 1, \ldots, n$,

$$\begin{aligned} X_i &= \delta + Z_i + \epsilon_{i1} \\ Y_i &= \phantom{\delta +{}} Z_i + \epsilon_{i2}, \text{ where} \end{aligned}$$

- $\delta$ is a constant.
- $E(Z_i) = \mu_z$, $Var(Z_i) = \sigma_z^2 > 0$
- $E(\epsilon_{i1}) = E(\epsilon_{i2}) = 0$,
- $Var(\epsilon_{i1}) = \sigma_1^2 > 0$, $Var(\epsilon_{i2}) = \sigma_2^2 > 0$
- $Z_i$ and $\epsilon_{ij}$ are all independent.

Prove $Cov(X_i, Y_i) > 0$.

$$\begin{aligned} Cov(X_i, Y_i) &= E(X_i Y_i) - E(X_i)E(Y_i) \\ &= E(X_i Y_i) - (\delta + \mu_z)\mu_z \text{ , and} \end{aligned}$$

$$\begin{aligned} E(X_i Y_i) &= E(\delta + Z_i + \varepsilon_{i1})(Z_i + \varepsilon_{i2}) \\ &= E(\delta Z_i + \delta \varepsilon_{i2} + Z_i^2 + Z_i \varepsilon_{i2} + \varepsilon_{i1} Z_i + \varepsilon_{i1} \varepsilon_{i2}) \\ &= \delta E(Z_i) + \delta E(\varepsilon_{i2}) + E(Z_i^2) + E(Z_i)E(\varepsilon_{i2}) + E(\varepsilon_{i1})E(Z_i) \\ &\quad + E(\varepsilon_{i1})E(\varepsilon_{i2}) \\ &= \delta \cdot \mu_z + \delta \cdot 0 + (\sigma_z^2 + \mu_z^2) + \mu_z \cdot 0 + 0 \cdot \mu_z + 0 \cdot 0 \\ &= \delta \mu_z + \sigma_z^2 + \mu_z^2 \end{aligned}$$

$$\begin{aligned} Cov(X_i, Y_i) &= \cancel{\delta \mu_z} + \sigma_z^2 + \cancel{\mu_z^2} \cancel{- \delta \mu_z} \cancel{- \mu_z^2} \\ &= \sigma_z^2 > 0 \end{aligned}$$

2. **Please slow down and read this question carefully**. Let $Y_1, \ldots, Y_n$ be a random sample from a distribution with expected value $\mu$ and variance $\sigma^2$. To test $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$, we will use the test statistic $Z_1 = \frac{\sqrt{n}(\overline{Y}-\mu_0)}{S}$, where $S$ is the sample standard deviation. The null hypothesis will be rejected if $Z_1 > z_\alpha$, where $z_\alpha$ is the point that cuts off the top $\alpha$ of the standard normal distribution. Notice this is a one-sided test.

   Also notice that by the Central Limit Theorem, $Z_2 = \frac{\sqrt{n}(\overline{Y}-\mu)}{\sigma}$ is approximately standard normal regardless of whether $H_0$ is true or not.

   (a) (4 points) **Give an expression for the approximate *power* of the test** — that is, the probability of rejecting $H_0$ when $H_0$ is false. Because this is a large sample test, assume $n$ is big enough so you can replace $S$ with $\sigma$ any time you wish, and probabilities remain roughly the same. Write your answer in terms of $\Phi$, the cumulative distribution function of a standard normal. The expression also involves $\mu$, $\sigma^2$, $z_\alpha$ and $n$. Show your work. Simplify. **Circle your final answer.**

$$\text{Power} = Pr\left\{ Z_1 > z_\alpha \right\} = Pr\left\{ \frac{\sqrt{n}(\overline{Y}-\mu_0)}{S} > z_\alpha \right\}$$

$$\approx Pr\left\{ \frac{\sqrt{n}(\overline{Y}-\mu_0)}{\sigma} > z_\alpha \right\} = Pr\left\{ \overline{Y}-\mu_0 > z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

$$= Pr\left\{ \overline{Y} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\} = Pr\left\{ \overline{Y}-\mu > \mu_0-\mu + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

$$= Pr\left\{ \frac{\sqrt{n}(\overline{Y}-\mu)}{\sigma} > \frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + \frac{\cancel{\sqrt{n}}}{\cancel{\sigma}} z_\alpha \frac{\cancel{\sigma}}{\cancel{\sqrt{n}}} \right\}$$

$$= Pr\left\{ Z_2 > \frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_\alpha \right\}$$

$$= 1 - \Phi\left( \frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_\alpha \right)$$

   (b) (1 point) When $\mu > \mu_0$, what happens to the power as the sample size $n \to \infty$? Use your answer to Question 2a. Show your work. Hint: The function $\Phi$ is continuous, so the limit of the function is the function of the limit.

$$\lim_{n\to\infty} \left( 1 - \Phi\left( \frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_\alpha \right) \right) = 1 - \Phi\left( \lim_{n\to\infty} \frac{\sqrt{n}(\mu_0-\mu)}{\sigma} + z_\alpha \right)$$

$$= 1 - \Phi(-\infty) = 1 - 0 = 1$$

Kind of rough but okay