Name Jerry

Student Number

STA 312 s2019 Quiz 5

Consider a random sample of right censored data from an exponential distribution with parameter $\lambda > 0$. In addition to the data $T_1, \ldots, T_n$, we have indicators $\delta_1, \ldots, \delta_n$, where $\delta_j = 1$ if T_j is a failure time, and $\delta_j = 0$ if T_j is a censoring time. You have already obtained the MLE $\widehat{\lambda} = \frac{\sum_{i=1}^n \delta_i}{\sum_{i=1}^n T_i}$ and its estimated asymptotic variance $\widehat{v}_n = \frac{\widehat{\lambda}^2}{\sum_{i=1}^n \delta_i}$. You may use these results without proof on this quiz.

1. (1 point) Give a formula for $\widehat{S}(t)$, the MLE of the survival function $S(t)$. Your answer is a formula that could be computed from the sample data for any given t.

$\hat{S}(t) = e^{-\hat{\lambda} t}$ where $\hat{\lambda}$ is given above

2. (3 points) Using the one-variable delta method (see formula sheet), derive a formula for the standard error of $\widehat{S}(t)$. Show your work. *Circle your final answer.*

$g(\lambda) = e^{-\lambda t}$, $g'(\lambda) = -t e^{-\lambda t}$, so

$g(\hat{\lambda}) = e^{-\hat{\lambda} t} \,\dot{\sim}\, N\left(e^{-\lambda t}, t^2 e^{-2\lambda t} \cdot \frac{\lambda^2}{\sum_{i=1}^n \delta_i}\right)$

And standard error is

$$\frac{\hat{\lambda} t e^{-\hat{\lambda} t}}{\sqrt{\sum_{i=1}^n \delta_i}}$$

No marks off for hats in the wrong places or missing – just a warning.

3. (1 point) Give a 95% confidence interval for the survival function $S(t)$. Your answer is two formulas, one for the lower confidence limit and one for the upper confidence limit. You cannot get credit for this question unless you have the standard error right.

lower: $e^{-\hat{\lambda}t} - 1.96 \dfrac{\hat{\lambda}t\, e^{-\hat{\lambda}t}}{\sqrt{\sum_{i=1}^{n} \delta_i}}$

upper: $e^{-\hat{\lambda}t} + 1.96 \dfrac{\hat{\lambda}t\, e^{-\hat{\lambda}t}}{\sqrt{\sum_{i=1}^{n} \delta_i}}$

4. In Question (2d) of Assignment 5, you obtained $\widehat{S}(t)$ and a 95% confidence interval for a sequence of t values. For $t = 1.0$, write the following numbers **in the spaces below**.

(a) (1 point) $\widehat{S}(1)$

0.1796

(b) (2 points) The lower confidence limit for $S(1)$.

0.0840

(c) (2 points) The upper confidence limit for $S(1)$.

0.2571

On your printout, circle the three numbers (one circle; they should be together) and write "Question 4" beside them.

Please attach the printout with your answers to Question 4 of this quiz (Question 2d of the assignment). Make sure your name and student number are written on the printout.

```
> ##########################################################################
> # Answer to computer part of Assignment 5
> ##########################################################################
> rm(list=ls()); options(scipen=999)
> exdata = read.table("http://www.utstat.utoronto.ca/~brunner/data/legal/expo.data2.txt")
> head(exdata); attach(exdata)
   Time Uncensored
1 0.179          0
2 1.024          1
3 0.189          1
4 0.345          1
5 0.977          1
6 0.241          1
>
>
> # 2a) MLE
> lambdahat = sum(Uncensored)/sum(Time);  lambdahat
[1] 1.717107
>
> # 2b Estimated asymptotic variance
> vhat = lambdahat^2 / sum(Uncensored);  vhat # Estimated asymptotic variance
[1] 0.07371138
> se = sqrt(vhat); se
[1] 0.2714984
>
> # 2c) 95% CI for lambda
> lower95 = lambdahat - 1.96*se; upper95 = lambdahat + 1.96*se
> c(lower95,upper95)
[1] 1.184970 2.249244
>
> # 2d) t, Shat(t), lower, upper
> t = seq(from=0,to=3,by=0.1)
> Shat = exp(-lambdahat*t)
> se_Shat = lambdahat*t*exp(-lambdahat*t)/sqrt(sum(Uncensored))
> Low1 = Shat - 1.96*se_Shat; High1 = Shat + 1.96*se_Shat
> cbind(t,Shat,Low1,High1)
        t        Shat          Low1      High1
 [1,] 0.0 1.000000000  1.0000000000 1.00000000
 [2,] 0.1 0.842222820  0.7974050386 0.88704060
 [3,] 0.2 0.709339279  0.6338461621 0.78483240
 [4,] 0.3 0.597421728  0.5020486893 0.69279477
 [5,] 0.4 0.503162213  0.3960617465 0.61026268
 [6,] 0.5 0.423774698  0.3110216269 0.53652777
 [7,] 0.6 0.356912721  0.2429568699 0.47086857
 [8,] 0.7 0.300600039  0.1886277838 0.41257229
 [9,] 0.8 0.253172212  0.1453943971 0.36095003
[10,] 0.9 0.213227415  0.1111078623 0.31534697
[11,] 1.0 0.179584995  0.0840211974 0.27514879    Question 4
[12,] 1.1 0.151250581  0.0627159687 0.23978519
[13,] 1.2 0.127386691  0.0460421046 0.20873128
[14,] 1.3 0.107287978  0.0330685223 0.18150743
[15,] 1.4 0.090360383  0.0230426549 0.15767811
[16,] 1.5 0.076103577  0.0153572979 0.13684986
[17,] 1.6 0.064096169  0.0095234732 0.11866887
[18,] 1.7 0.053983256  0.0051482384 0.10281827
[19,] 1.8 0.045465930  0.0019165540 0.08901531
[20,] 1.9 0.038292444 -0.0004235166 0.07700840
[21,] 2.0 0.032250770 -0.0020728777 0.06657442
[22,] 2.1 0.027162335 -0.0031912329 0.05751590
[23,] 2.2 0.022876738 -0.0039050848 0.04965856
[24,] 2.3 0.019267311 -0.0043142362 0.04284886
[25,] 2.4 0.016227369 -0.0044970663 0.03695180
[26,] 2.5 0.013667060 -0.0045148065 0.03184893
[27,] 2.6 0.011510710 -0.0044150004 0.02743642
[28,] 2.7 0.009694583 -0.0042342986 0.02362346
[29,] 2.8 0.008164999 -0.0040007126 0.02033071
[30,] 2.9 0.006876748 -0.0037354286 0.01748893
[31,] 3.0 0.005791754 -0.0034542638 0.01503777
```

```
>
> # 2e Another CI
> me2 = 1.96*lambdahat*t/sqrt(sum(Uncensored))
> Low2 = exp(-lambdahat*t-me2); High2 =  exp(-lambdahat*t+me2)
> cbind(t,Shat,Low2,High2)
        t        Shat        Low2      High2
 [1,] 0.0 1.000000000 1.000000000 1.00000000
 [2,] 0.1 0.842222820 0.798576625 0.88825450
 [3,] 0.2 0.709339279 0.637724626 0.78899605
 [4,] 0.3 0.597421728 0.509271980 0.70082929
 [5,] 0.4 0.503162213 0.406692699 0.62251477
 [6,] 0.5 0.423774698 0.324775283 0.55295155
 [7,] 0.6 0.356912721 0.259357949 0.49116170
 [8,] 0.7 0.300600039 0.207117196 0.43627659
 [9,] 0.8 0.253172212 0.165398951 0.38752464
[10,] 0.9 0.213227415 0.132083736 0.34422051
[11,] 1.0 0.179584995 0.105478984 0.30575541
[12,] 1.1 0.151250581 0.084233051 0.27158862
[13,] 1.2 0.127386691 0.067266546 0.24123981
[14,] 1.3 0.107287978 0.053717491 0.21428235
[15,] 1.4 0.090360383 0.042897533 0.19033726
[16,] 1.5 0.076103577 0.034256967 0.16906793
[17,] 1.6 0.064096169 0.027356813 0.15017535
[18,] 1.7 0.053983256 0.021846511 0.13339393
[19,] 1.8 0.045465930 0.017446113 0.11848776
[20,] 1.9 0.038292444 0.013932058 0.10524728
[21,] 2.0 0.032250770 0.011125816 0.09348637
[22,] 2.1 0.027162335 0.008884817 0.08303969
[23,] 2.2 0.022876738 0.007095207 0.07376038
[24,] 2.3 0.019267311 0.005666066 0.06551799
[25,] 2.4 0.016227369 0.004524788 0.05819665
[26,] 2.5 0.013667060 0.003613390 0.05169343
[27,] 2.6 0.011510710 0.002885569 0.04591693
[28,] 2.7 0.009694583 0.002304348 0.04078592
[29,] 2.8 0.008164999 0.001840198 0.03622827
[30,] 2.9 0.006876748 0.001469539 0.03217993
[31,] 3.0 0.005791754 0.001173540 0.02858396
>
> High1-Low1-High2+Low2 # Traditional CI is consistently narrower.
 [1]  0.00000000000 -0.00004230944 -0.00028519293 -0.00081123554 -0.00162114019
 [6] -0.00267012102 -0.00389204572 -0.00521488146 -0.00657005910 -0.00789766404
[11] -0.00914883327 -0.01028634487 -0.01128409536 -0.01212594693 -0.01280427099
[16] -0.01331840286 -0.01367314243 -0.01387738031 -0.01394289016 -0.01388330265
[21] -0.01371325981 -0.01344773846 -0.01310152574 -0.01268882739 -0.01222298886
[26] -0.01171630989 -0.01117993511 -0.01062380468 -0.01005665147 -0.00948603291
[31] -0.00891838797
>
> # 3a) Numerical MLE
>
> mloglike = function(lambda,t,delta)
+     { lambda*sum(t)-log(lambda)*sum(delta) }
> search = optim(par=1, fn=mloglike, t=Time,delta=Uncensored,
+                       hessian=TRUE, lower=0, method='L-BFGS-B')
> search
$par
[1] 1.717107

$value
[1] 18.37437

$counts
function gradient
       7        7

$convergence
[1] 0

$message
```

```
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"

$hessian
         [,1]
[1,] 13.56643

> c(lambdahat,search$par)
[1] 1.717107 1.717107
>
> # 3b) Numerical estimated asymptotic variance
> vhat2 = 1/search$hessian
> c(vhat,vhat2)
[1] 0.07371138 0.07371135
>
```