STA 312 f22 Assignment 3

1. (a) $n_j \sim \text{Binomial}(n, \pi_j)$

   (b) $E(n_j) = n\pi_j$

   (c) $\hat{\mu}_j = n\hat{\pi}_j$

2. $$G^2 = -2 \log \frac{\max_{\beta \in \mathcal{B}_0} \ell(\beta)}{\max_{\beta \in \mathcal{B}} \ell(\beta)} = -2 \log \frac{\ell(\hat{\underset{\sim}{\pi}})}{\ell(\underset{\sim}{p})}$$

$$= 2 \log \left(\frac{\ell(\hat{\underset{\sim}{\pi}})}{\ell(\underset{\sim}{p})} \right)^{-1} = 2 \log \frac{\ell(\underset{\sim}{p})}{\ell(\hat{\underset{\sim}{\pi}})}$$

$$= 2 \log \frac{p_1^{n_1} p_2^{n_2} \cdots p_c^{n_c}}{\hat{\pi}_1^{n_1} \hat{\pi}_2^{n_2} \cdots \hat{\pi}_c^{n_c}} = 2 \log \left(\left(\frac{p_1}{\hat{\pi}_1}\right)^{n_1} \cdots \left(\frac{p_c}{\hat{\pi}_c}\right)^{n_c} \right)$$

$$= 2 \sum_{j=1}^{c} n_j \log \frac{p_j}{\hat{\pi}_j} = 2 \sum_{j=1}^{c} n_j \log \frac{n_j}{n\hat{\pi}_j}$$

$$= 2 \sum_{j=1}^{c} n_j \log \frac{n_j}{\hat{\mu}_j}$$

(3)

(a) Noting that $\mathcal{B}_0$ has only the single point π_0, $\hat{\pi}_1 = \pi_0$ and $\hat{\mu}_1 = n\pi_0$. Then,

$$G^2 = 2\sum_{j=1}^{c} n_j \log\frac{n_j}{\hat{\mu}_j} = 2\left(n_1 \log\frac{n_1}{n\pi_0} + n_2 \log\frac{n_2}{n(1-\pi_0)}\right)$$

or $$2n\left(\overset{p}{\frac{n_1}{n}} \log\frac{p}{\pi_0} + \overset{1-p}{\frac{n_2}{n}} \log\frac{(1-p)}{(1-\pi_0)}\right)$$

(b) $$X^2 = \sum_{j=1}^{c}\frac{(n_j-\hat{\mu}_j)^2}{\hat{\mu}_j} = \frac{(n_1-\pi_0 n)^2}{\pi_0 n} + \frac{(n_2-(1-\pi_0)n)^2}{(1-\pi_0)n}$$

$$= \frac{[n(p-\pi_0)]^2}{n\pi_0} + \frac{[n(1-p-(1-\pi_0))]^2}{n(1-\pi_0)}$$

$$= \frac{n^{\cancel{2}}(p-\pi_0)^2}{\cancel{n}\pi_0} + \frac{n^{\cancel{2}}(\cancel{1}-p\cancel{-1}+\pi_0)^2}{\cancel{n}(1-\pi_0)}$$

$$= n\left[\frac{(p-\pi_0)^2}{\pi_0} + \frac{(p-\pi_0)^2}{1-\pi_0}\right] = n(p-\pi_0)^2\cdot\left(\frac{1}{\pi_0}+\frac{1}{1-\pi_0}\right)$$

$$= n(p-\pi_0)^2\cdot\frac{1\cancel{-\pi_0+\pi_0}}{\pi_0(1-\pi_0)} = \frac{n(p-\pi_0)^2}{\pi_0(1-\pi_0)} = z_1^2$$

(c) df = 1

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> # R work for Assignment 3
>
> # Beer 1 (Question 5)
> obs = c(30,24,22,28,9,7); obs
[1] 30 24 22 28  9  7
> ex = 120*c(1,1,1,1,1,1)/6 ; ex
[1] 20 20 20 20 20 20
> G2 = 2*sum(obs*log(obs/ex)); G2
[1] 27.04479
> pvalG <- 1-pchisq(G2,5); pvalG
[1] 5.590826e-05
> X2 = sum((obs-ex)^2/ex); X2
[1] 23.7
> pvalX <- 1-pchisq(X2,5); pvalX
[1] 0.0002479085
>
>
>
> # Beer 2 (R part of Question 6)
> expctd = c(26,26,26,26,8,8)
> G2 = 2*sum(obs*log(obs/expctd)); G2
[1] 1.794323
> X2 = sum((obs-expctd)^2/expctd); X2
[1] 1.788462
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(6) (a) $H_0: \pi_1 = \pi_2 = \pi_3 = \pi_4$, <u>and</u> $\pi_5 = \pi_6$,

noting $\pi_6 = 1 - \pi_1 - \pi_2 - \pi_3 - \pi_4 - \pi_5$

(b) $df = 4$

(c) It looks like There are 2 parameters under H_0, π_1 & π_5

$$l_0 = \pi_1^{n_1+n_2+n_3+n_4} \pi_5^{n_5} (1 - 4\pi_1 - \pi_5)^{n_6},$$

but since $\pi_5 = \pi_6$,

$$\pi_5 = 1 - 4\pi_1 - \pi_5 \Rightarrow 2\pi_5 = 1 - 4\pi_1$$

$$\Rightarrow \pi_5 = \tfrac{1}{2} - 2\pi_1 \text{ and}$$

$$l_0 = \pi_1^{n_1+n_2+n_3+n_4} \left(\tfrac{1}{2} - 2\pi_1\right)^{n_5+n_6}$$

This is nice because the parameter space is 5-d, and H_0 imposes 4 constraints - so there is one free parameter.

$$\frac{\partial}{\partial \pi_1} \log l_0 = \frac{\partial}{\partial \pi_1}\left((n_1+n_2+n_3+n_4)\log \pi_1 + (n_5+n_6)\log\left(\tfrac{1}{2} - 2\pi_1\right)\right)$$

$$= \frac{n_1+n_2+n_3+n_4}{\pi_1} - \frac{2(n_5+n_6)}{\frac{1}{2} - 2\pi_1} \overset{set}{=} 0$$

$$\Rightarrow 2\pi_1(n_5+n_6) = \tfrac{1}{2}(n_1+n_2+n_3+n_4) - 2\pi_1(n_1+n_2+n_3+n_4)$$

$$\Rightarrow 2\pi_1(n_1+n_2+n_3+n_4+n_5+n_6) = \tfrac{1}{2}(n_1+n_2+n_3+n_4)$$

$$\Rightarrow \hat{\pi}_1 = \frac{1}{4}\,\frac{n_1+n_2+n_3+n_4}{n} = \hat{\pi}_2 = \hat{\pi}_3 = \hat{\pi}_4 \text{ and}$$

$$\hat{\pi}_5 = \tfrac{1}{2} - 2\hat{\pi}_1 = \frac{n}{2n} - \frac{n_1+n_2+n_3+n_4}{2n} = \frac{1}{2}\,\frac{n_5+n_6}{n} = \hat{\pi}_6$$

6. cont

$$\frac{104}{\frac{16}{120}}$$

(d) $\vec{\pi} = \left(\frac{n_1+n_2+n_3+n_4}{4n}, \text{same}, \text{same}, \text{same}, \frac{n_5+n_6}{2n}, \frac{n_5+n_6}{2n} \right)$

(e) $\left(\frac{26}{120}, \frac{26}{120}, \frac{26}{120}, \frac{26}{120}, \frac{8}{120}, \frac{8}{120}\right) = (0.217, 0.217, 0.217, 0.217, 0.067, 0.067)$

$(26, 26, 26, 26, 8, 8) = \vec{\mu}$

(f) $G^2 = 1.794$ with R

(g) $\chi^2 = 1.788$ with R

(h) Crit value with 4 df $= 9.488$

(i) No

(j) No

(k) There is no evidence that pref for the leagues is different or that pref for the coles is different

(7) (a) $H_0: \cancel{\pi_1} + \pi_3 = \cancel{\pi_1} + \pi_2 \Leftrightarrow \pi_2 = \pi_3$

(b) $df = 1$

(c) $l_0 = \pi_1^{n_1} \pi_2^{n_2} \pi_2^{n_3} (1 - \pi_1 - 2\pi_2)^{n_4}$

$= \pi_1^{n_1} \pi_2^{n_2+n_3} (1 - \pi_1 - 2\pi_2)^{n_4}$

$$\frac{\partial}{\partial \pi_1} \log l_0 = \frac{\partial}{\partial \pi_1} \left(n_1 \log \pi_1 + (n_2+n_3) \log \pi_2 + n_4 \log (1 - \pi_1 - 2\pi_2) \right)$$

$$= \frac{n_1}{\pi_1} - \frac{n_4}{1 - \pi_1 - 2\pi_2} \stackrel{set}{=} 0$$

$\Rightarrow n_4 \pi_1 = n_1 - n_1 \pi_1 - 2 n_1 \pi_2 \Rightarrow \pi_1 (n_1 + n_4) = n_1 (1 - 2\pi_2)$

$$\Rightarrow \pi_1 = \frac{n_1 (1 - 2\pi_2)}{n_1 + n_4}$$

$$\frac{\partial}{\partial \pi_2} \log l_0 = \frac{n_2 + n_3}{\pi_2} - \frac{2 n_4}{1 - \pi_1 - 2\pi_2} \stackrel{set}{=} 0$$

$\Leftrightarrow 2 n_4 \pi_2 = (n_2 + n_3)(1 - \pi_1) - 2(n_2 + n_3) \pi_2$

$\Leftrightarrow \pi_2 (2n_4 + 2n_2 + 2n_3) = (n_2 + n_3)(1 - \pi_1)$

$$\Leftrightarrow \pi_2 = \frac{n_2 + n_3}{2(n_2 + n_3 + n_4)} (1 - \pi_1)$$

7c continued Have $\pi_1 = \dfrac{n_1(1-2\pi_2)}{n_1+n_4}$ and

$$\pi_2 = \frac{n_2+n_3}{2(n_2+n_3+n_4)}(1-\pi_1)$$

$$= \frac{n_2+n_3}{2(n_2+n_3+n_4)}\left(\frac{\cancel{n_1}+n_4-\cancel{n_1}+2\pi_2 n_1}{n_1+n_4}\right)$$

hmm...