routines are included in the collection of EXCEL routines available at the ACTEX website.

When there is some systematic way in which the payments of an annuity vary, it may be possible to algebraically simplify the present or accumulated value. Retirement annuities often have a "cost-of-living" increase or inflation-adjustment provision. This means that the annuity payment is adjusted periodically (usually annually) at a rate related to some measure of the change in the inflation rate. Often the payment adjustment is related to a "consumer price index," or some other standard measure of price inflation. Algebraically, an annuity whose payments are adjusted according to inflation would have payments that tend to increase geometrically. Inflation is unlikely to be constant from year to year, but if we consider a simplified situation where there is a level rate of inflation every year, say r, then an inflation adjusted annuity would have payments which grow by a factor of $1+r$ each year. Many insurance companies sell indexed annuities with a fixed index rate r. The following example illustrates this idea.

EXAMPLE 2.17

(*Annuity whose payments form a geometric progression*)
Smith wishes to purchase a 20-year annuity with annual payments beginning one year from now. The annuity will be valued at an effective annual rate of 11%. Smith anticipates an effective annual inflation rate over the next 20 years of 4% per year, so he wants to index his annuity at a 4% rate. In other words he would like each payment after the first to be 4% larger than the previous one. If Smith's first payment is to be 26,000, what is the present value of the annuity?

SOLUTION

The series of payments is

$$26{,}000;\ 26{,}000(1.04);\ 26{,}000(1.04)^2;\ldots;\ 26{,}000(1.04)^{19},$$

and has present value

$$26{,}000\cdot v_{.11}+26{,}000(1.04)\cdot v_{.11}^2+26{,}000(1.04)^2\cdot v_{.11}^3+\cdots +26{,}000(1.04)^{19}\cdot v_{.11}^{20}.$$

This can be written as $26{,}000 \cdot v\left[1+1.04v+(1.04v)^2+\cdots+(1.04v)^{19}\right]$, which then simplifies to $26{,}000 \cdot v\left[\frac{1-(1.04v)^{20}}{1-1.04v}\right] = 270{,}484.$ □

The important point to note in Example 2.17 is that when payments form a geometric progression, the ratio in the geometric progression combines with the present value factor so that the present value of the annuity reduces to another geometric progression. Another way of viewing an annuity whose payments form a geometric progression is illustrated in the next example.

EXAMPLE 2.18 (*Geometric progression*)

A series of n periodic payments has first payment of amount 1, and all subsequent payments are $(1+r)$ times the size of the previous payment. At a rate of interest i per payment period, show that the present value of the series at the time of the first payment can be written as $\ddot{a}_{\overline{n}|j}$ for an appropriately defined interest rate j.

SOLUTION

The present value of the series at the time of the first payment is

$$1+(1+r)v_i+\left[(1+r)v_i\right]^2+\cdots+\left[(1+r)v_i\right]^{n-1}$$

$$= \frac{1-\left[(1+r)v_i\right]^n}{1-(1+r)v_i} = \frac{1-\left(\frac{1+r}{1+i}\right)^n}{1-\frac{1+r}{1+i}}.$$

We want this to be $\ddot{a}_{\overline{n}|j} = \frac{1-v_j^n}{1-v_j}$. If we let $v_j = \frac{1+r}{1+i}$, then the present value will be of the proper form. But if $\frac{1}{1+j} = v_j = \frac{1+r}{1+i}$, then $1+j = \frac{1+i}{1+r}$, so that $j = \frac{i-r}{1+r}$. □

We see from Example 2.18 that the present value of an annuity whose payments form a geometric progression can be formulated as an annuity with level payments valued at a modified rate of interest (an *inflation-adjusted* rate