Bounds On The Tails Of Convolutions Of Compound Distributions

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Abstract

In this paper the convolution of a compound distribution with another given ditribution is considered. Upper and lower bounds on the tail probabilities of this type of distribution are derived. Applications to the ruin probabilities of compound Poisson claims processes perturbed by diffusion and to an approximation to the equilibrium waiting time distribution of the M/G/c queue are given.

Keywords: Compound distributions; ruin probabilities; M/G/c queue

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1 Introduction

Let $Y_i, i = 1, 2, \cdots$ be a sequence of i.i.d. positive random variables with distribution function $F(y)$ and N be a counting random variable independent of $\boldsymbol{Y}_i$, with

$$Pr(N = n) = p_n, \quad n = 0, 1, 2, \cdots. \tag{1.1}$$

Also let

$$X = Y_1 + Y_2 + \cdots + Y_N. \tag{1.2}$$

Thus, the distribution function $G(x)$ of X is

$$G(x) = \sum_{n=0}^{\infty} p_n F^{*n}(x), \tag{1.3}$$

where $F^{*n}(x)$ is the distribution function of the n-fold convolution.

In this paper we consider bounds for the tails of the convolution of $G(x)$ with a distribution function $A(x)$, namely,

$$\psi(x) = \int_x^{\infty} dA * G(y) = \overline{A}(x) + \int_0^x \overline{G}(x-y)dA(y), \tag{1.4}$$

where $\overline{A}(x) = 1 - A(x)$, and $\overline{G}(x) = 1 - G(x)$.

The convolution of a compound distribution with another given distribution is of central interest in many disciplines. In queueing theory, an approximation to the equilibrium waiting time distribution of arriving customers for the M/G/c queue and for many variants of the M/G/1 queue can be expressed in this form(van Hoorn, 1984; Tijms, 1986; Neuts, 1986). In risk theory, when a diffusion process is added to the compound Poisson claims process the probability of infinite ruin is the tail probability of the convolution of a compound geometric distribution with an exponential distribution(Dufresne and Gerber, 1991; Gerber, 1970).

Recently, Willmot and Lin(Willmot and Lin, 1994; Willmot, 1994; Lin, 1996) have obtained upper and lower bounds for compound distributions that generalize the classical Cramer-Lundberg upper bound for ruin probabilities.

Suppose $\phi < 1$ and the distribution function $B(x)$ satisfies

$$\int_0^\infty \{\overline{B}(y)\}^{-1} dF(y) = \phi^{-1}. \tag{1.5}$$

where $\overline{B}(x) = 1 - B(x)$.

Proposition 1.1(Willmot, 1994) If

(i)

$$a_n \leq \phi a_{n-1}, \quad n = 1, 2, \cdots, \tag{1.6}$$

where $a_n = \sum_{m=n+1}^{\infty} p_m$;

(ii) $B(x)$ is NWU(new worse than used, i.e. $\overline{B}(x)\overline{B}(y) \leq \overline{B}(x+y)$, for $x > 0$, $y > 0$);

(iii) Equation (1.5) holds;

(iv)

$$[c(x)]^{-1} \leq \inf_{0 \leq z \leq x, \overline{F}(z) > 0} \frac{\overline{B}(z) \int_z^\infty [\overline{B}(y)]^{-1} dF(y)}{\overline{F}(z)}, \tag{1.7}$$

then

$$\overline{G}(x) \leq \frac{1 - p_0}{\phi} c(x) \overline{B}(x), \qquad x \geq 0\ . \tag{1.8}$$

Proposition 1.2(Lin, 1996) If conditions (i), (ii) and (iii) in Proposition 1.1 hold and

$$[\Delta(x)]^{-1} \leq \inf_{0 \leq z \leq x, \overline{F}(z) > 0} \frac{\int_z^\infty [\overline{B}(x - z + y)]^{-1} dF(y)}{\overline{F}(z)}, \tag{1.9}$$

then

$$\psi(x) \leq \frac{1 - p_0}{\phi} \Delta(x). \tag{1.10}$$

Remark: The conditions in the above propositions are slightly different but are the same when $\overline{B}(x) = e^{-\kappa x}$.

Proposition 1.3(Lin, 1996) If

(i)

$$a_n \geq \phi a_{n-1}, \quad n = 1, 2, \cdots; \tag{1.11}$$

(ii) $B(x)$ is NBU(new better than used, i.e. $\overline{B}(x)\overline{B}(y) \geq \overline{B}(x+y)$, for $x > 0$, $y > 0$);

(iii) Equation (1.5) holds;

(iv)

$$[\Delta(x)]^{-1} \geq \sup_{0 \leq z \leq x, \overline{F}(z) > 0} \frac{\int_z^\infty [\overline{B}(x - z + y)]^{-1} dF(y)}{\overline{F}(z)}, \tag{1.12}$$

then

$$\psi(x) \geq \frac{1 - p_0}{\phi} \Delta(x). \tag{1.13}$$

In this paper, we extend these results to the tail of the convolution of a compound distribution with an arbitrary distribution. We then apply our results to the ruin probabilities of a compound Poisson claims process perturbed by diffusion and to the equilibrium waiting time distribution of the M/G/c queue.

2 Bounds for the Tail of Convolutions

In this section we derive our main results. The following theorem gives an upper bound of the tail $\psi(x)$ defined in (1.4).

Theorem 2.1 Assume that

(i) The conditions of Proposition 1.1 hold;

(ii) The quantity ϕ_A is defined by

$$\phi_A^{-1} = \int_0^\infty \{\overline{B}(y)\}^{-1} dA(y); \tag{2.1}$$

(iii) The function $c_A(x)$ satisfies

$$[c_A(x)]^{-1} \leq \frac{\overline{B}(x)\int_x^\infty [\overline{B}(y)]^{-1}dA(y)}{\overline{A}(x)}, \quad x \geq 0. \tag{2.2}$$

Then,

$$\psi(x) \leq \{1 - \frac{1-p_0}{\phi}\frac{c(x)}{c_A(x)}\}\overline{A}(x) + \{\frac{1-p_0}{\phi\phi_A}c(x)\}\overline{B}(x). \tag{2.3}$$

Proof:

$$\begin{aligned}
\psi(x) &= \overline{A}(x) + \int_0^x \overline{G}(x-y)dA(y) \\
&\leq \overline{A}(x) + \frac{1-p_0}{\phi}\int_0^x c(x-y)\overline{B}(x-y)dA(y) \\
&\leq \overline{A}(x) + \frac{1-p_0}{\phi}c(x)\overline{B}(x)\int_0^x \{\overline{B}(y)\}^{-1}dA(y) \\
&= \overline{A}(x) + \frac{1-p_0}{\phi}c(x)\overline{B}(x)\{\frac{1}{\phi_A} - \int_x^\infty \{\overline{B}(y)\}^{-1}dA(y)\} \\
&= \overline{A}(x) + \frac{1-p_0}{\phi\phi_A}c(x)\overline{B}(x) - \frac{1-p_0}{\phi}c(x)\overline{B}(x)\int_x^\infty \{\overline{B}(y)\}^{-1}dA(y) \\
&\leq \overline{A}(x) + \frac{1-p_0}{\phi\phi_A}c(x)\overline{B}(x) - \frac{1-p_0}{\phi}\frac{c(x)}{c_A(x)}\overline{A}(x) \\
&= \{1 - \frac{1-p_0}{\phi}\frac{c(x)}{c_A(x)}\}\overline{A}(x) + \{\frac{1-p_0}{\phi\phi_A}c(x)\}\overline{B}(x).
\end{aligned} \tag{2.4}$$

$\square$

Some remarks are now made.

(a) In many applications, we are able to choose $c_A(x) = \phi_A$, for example, if $\overline{B}(x) = e^{-\kappa x}$ and $A(x)$ is NWUC(see Willmot, 1994). In this case, the bound in Theorem 2.1 looks like a mixture.

(b) If the moment generating function exists for both $F(x)$ and $A(x)$,we may choose $\overline{B}(x)$ to be an exponential distribution. If not, a Pareto distribution $\overline{B}(x) = (1+\kappa x)^{-m}$ will be a good candidate(see the discussions in Willmot(1994), Lin(1996) and those in the next section).

In order to find a lower bound for the convolution type we have considered, we assume that both $F(x)$ and $A(x)$ have a moment generating function without loss of generality. Otherwise, we consider the distributions obtained by truncating $F(x)$ and $A(x)$ from above and the lower bound derived in this way will be also the lower bound for the original distribution(see Lin, 1996 for details). Thus, we choose $\overline{B}(x) = e^{-\kappa x}$. If the conditions in Proposition 1.3 are satisfied, then

$$\psi(x) \geq \frac{1-p_0}{\phi}\delta(x)e^{-\kappa x}, \tag{2.5}$$

where

$$\delta^{-1}(x) \geq \sup_{0 \leq z \leq x, \overline{F}(z)>0} \frac{\int_z^\infty e^{\kappa y} dF(y)}{e^{\kappa z}\overline{F}(z)}. \tag{2.6}$$

Theorem 2.2 Assume that

(i) the conditions of Proposition 1.3 hold with $\overline{B}(x) = e^{-\kappa x}$;

(ii) The quantity ϕ_A is defined by

$$\phi_A^{-1} = \int_0^\infty e^{\kappa y} dA(y); \tag{2.7}$$

(iii) The function $c_A(x)$ satisfies

$$[c_A(x)]^{-1} \geq \frac{e^{-\kappa x}\int_x^\infty e^{\kappa y} dA(y)}{\overline{A}(x)}, \quad x \geq 0. \tag{2.8}$$

Then,

$$\psi(x) \geq \{1 - \frac{1-p_0}{\phi}\frac{\delta(x)}{c_A(x)}\}\overline{A}(x) + \{\frac{1-p_0}{\phi\phi_A}\delta(x)\}e^{-\kappa x}. \tag{2.9}$$

Proof:

$$\begin{aligned}
\psi(x) &= \overline{A}(x) + \int_0^x \overline{G}(x-y)dA(y) \\
&\geq \overline{A}(x) + \frac{1-p_0}{\phi}\int_0^x \delta(x-y)e^{-\kappa(x-y)}dA(y)
\end{aligned}$$

$$
\begin{aligned}
&\geq \overline{A}(x) + \frac{1-p_0}{\phi}\delta(x)e^{-\kappa x}\int_0^x e^{\kappa y}dA(y) \\
&= \overline{A}(x) + \frac{1-p_0}{\phi}\delta(x)e^{-\kappa x}\{\frac{1}{\phi_A} - \int_x^\infty e^{\kappa y}dA(y)\} \\
&\geq \overline{A}(x) + \frac{1-p_0}{\phi\phi_A}\delta(x)e^{-\kappa x} - \frac{1-p_0}{\phi}\frac{\delta(x)}{c_A(x)}\overline{A}(x) \\
&= \{1 - \frac{1-p_0}{\phi}\frac{\delta(x)}{c_A(x)}\}\overline{A}(x) + \{\frac{1-p_0}{\phi\phi_A}\delta(x)\}e^{-\kappa x}. \qquad (2.10)
\end{aligned}
$$

$\square$

Similar to the remark (a) following Theorem 2.1, if $\overline{B}(x) = e^{-\kappa x}$ and $A(x)$ is NBUC, then $c_A(x) = \phi_A$(also see Lin, 1996).

3 Applications

In this section, we apply our results to two cases: the compound Poisson claims process perturbed by diffusion and the M/G/c queue.

Dufresne and Gerber(1991)(also see Gerber, 1970; Veraverbeke, 1993; Furrer and Schmidli, 1994) considered the following model:

$$U(t) = x + ct - S(t) + W(t), \quad t \geq 0, \qquad (3.1)$$

where $U(t)$ is the surplus at time t, $x \geq 0$ is the initial surplus, c is the premium rate, $S(t)$ is the aggregate claims process which is assumed to be compound Poisson with parameter λ and individual claim distribution $P(y)$, and $W(t)$ is a Wiener process with infinitesimal drift 0 and infinitesimal variance $2D$. It is also assumed that $c > \lambda\mu$, where μ is the expected claim size. Thus the relative security loading is $q = (c - \lambda\mu)/c$.

Let $R(x)$ denote the probability of survival, i.e.

$$R(x) = \Pr\{U(t) \geq 0 \text{ for all } t \geq 0\} \qquad (3.2)$$

and $\psi(x)$ denote the probability of infinite ruin. Then, $\psi(x) = 1 - R(x)$.

Dufresne and Gerber(1991) have shown that

$$R(x) = \sum_{n=0}^{\infty} q(1-q)^n H_1^{*(n+1)} * H_2^{*n}(x), \tag{3.3}$$

where H_1 is the exponential distribution with parameter $\xi = c/D$ and H_2 has the density function $\frac{1-P(x)}{\mu}$.

They further showed that if the moment generating function of P exists and there is a positive κ satisfying

$$\lambda \int_0^{\infty} e^{\kappa x} dP(x) + D\kappa^2 = \lambda + c\kappa, \tag{3.4}$$

then

$$\psi(x) \leq e^{-\kappa x}. \tag{3.5}$$

Obviously, if we let $A(x) = H_1(x)$ and

$$G(x) = \sum_{n=0}^{\infty} q(1-q)^n H_1^{*n} * H_2^{*n}(x),$$

then $\psi(x)$ is the tail of the convolution of a compound geometric distribution with an exponential distribution and our results may apply. We will show in the following that the exponential bound in (3.5) can be refined in some cases and a lower exponential bound with the same κ can be obtained. We will also show how to obtain an upper bound when the generating function of P does not exist.

Choose $\overline{B}(x) = e^{-\kappa x}$ with $\kappa > 0$ satisfying (3.4). It can be shown that

$$\int_0^{\infty} e^{\kappa x} dH_1 * H_2(x) = (1-q)^{-1}. \tag{3.6}$$

It is also easy to see that $\phi = 1-q$, $\phi_A = c_A(x) = \frac{\xi-\kappa}{\xi}$. Hence,

$$\psi(x) \leq \{1 - \frac{c(x)}{\phi_A}\}e^{-\xi x} + \frac{c(x)}{\phi_A}e^{-\kappa x}. \tag{3.7}$$

where

$$[c(x)]^{-1} = \inf_{0\leq z\leq x} \frac{\int_z^{\infty} e^{\kappa y} dH_1 * H_2(y)}{e^{\kappa z}\overline{H_1 * H_2}(z)}. \tag{3.8}$$

Since $\kappa < \xi$, if $\frac{c(x)}{\phi_A} < 1$ this upper bound is a refinement of (3.5). That is the case when $P(x)$ is an exponential distribution.

Similarly, a lower bound can be obtained by replacing $c(x)$ with $\delta(x)$ and all other parameters remain the same, i.e.

$$\psi(x) \geq \{1 - \frac{\delta(x)}{\phi_A}\}e^{-\xi x} + \frac{\delta(x)}{\phi_A}e^{-\kappa x}, \tag{3.9}$$

where

$$[\delta(x)]^{-1} = \sup_{0 \leq z \leq x} \frac{\int_z^\infty e^{\kappa y} dH_1 * H_2(y)}{e^{\kappa z}\overline{H_1 * H_2}(z)}. \tag{3.10}$$

We now deal with the case that $P(x)$ has a finite number of moments(this class contains logarithmic and subexponential distributions).

Suppose that $P(x)$ has the first $m+1$ moments. We choose $\overline{B}(x) = (1+\kappa x)^{-m}$ with κ satisfying

$$\int_0^\infty (1+\kappa x)^m dH_1 * H_2(x) = (1-q)^{-1}. \tag{3.11}$$

It can be shown that κ is the largest real root of the following polynomial of degree m:

$$\lambda \int_0^\infty [1 - P(y)]\{\sum_{i=0}^{m}(m)_i(\frac{\kappa}{\xi})^i(1+\kappa y)^{m-i}\}dy = c, \tag{3.12}$$

whose coefficients are a linear combination of the moments of $P(x)$, where $(m)_i = m(m-1)\cdots(m-i+1); (m)_0 = 1$.

$$\phi_A^{-1} = \sum_{i=0}^{m}(m)_i(\frac{\kappa}{\xi})^i. \tag{3.13}$$

Therefore,

$$\psi(x) \leq \{1 - \frac{c(x)}{c_A(x)}\}e^{-\xi x} + \frac{c(x)}{\phi_A}(1+\kappa x)^{-m}, \tag{3.14}$$

where

$$[c(x)]^{-1} = \inf_{0 \leq z \leq x} \frac{\int_z^\infty (1+\kappa y)^m dH_1 * H_2(y)}{(1+\kappa z)^m \overline{H_1 * H_2}(z)}. \tag{3.15}$$

We now consider bounds on the tail of the equilibrium waiting time distribution of the M/G/c queue. Assume that the distribution for the number of arrivals is Poisson with rate λ and the service time distribution is $P(x)$ with mean μ. Let $\rho = \lambda\mu/c$ be the traffic intensity, where $\rho < 1$. The approximate equilibrium waiting time distribution function $W_1(x)$ of the M/G/c queue is then expressed as:

$$W_1(x) = A * W(x), \tag{3.16}$$

where $A(x) = 1-[\overline{H}(x)]^c$ and $H(x)$ has the density function $\frac{1-P(x)}{\mu}$, and $W(x)$ is compound geometric with $p_n = (1-\rho)\rho^n$ and Y_i has the distribution function $H(cx)$.

Thus, we have

$$\int_0^\infty e^{\kappa x} dH(cx) = \rho^{-1}, \tag{3.17}$$

$$\rho_A^{-1} = \int_0^\infty e^{\kappa x} d\{1 - [\overline{H}(x)]^c\}, \tag{3.18}$$

and

$$c_A(x)^{-1} = \frac{\int_x^\infty e^{\kappa y} d\{1 - [\overline{H}(x)]^c\}}{e^{\kappa x}[\overline{H}(x)]^c}. \tag{3.19}$$

Thus,

$$\overline{W_1}(x) \leq \{1 - \frac{c(x)}{c_A(x)}\}[\overline{H}(x)]^c + \frac{c(x)}{\rho_A} e^{-\kappa x}. \tag{3.20}$$

If $P(x)$ is IMRL, a simpler bound can be obtained. In this case, we can choose $c(x) = \rho$ and $c_A(x) = \rho_A$, and

$$\overline{W_1}(x) \leq \{1 - \frac{\rho}{\rho_A}\}[\overline{H}(x)]^c + \frac{\rho}{\rho_A} e^{-\kappa x}. \tag{3.21}$$

It follows from $[\overline{H}(x)]^c \leq \overline{H}(cx)$ that $\rho \leq \rho_A$. Hence (3.21) is a mixture. Moreover, a different type of bound can also be chosen. Let $\overline{B}(x) = [\overline{H}(cx)]^{1-\rho}$. Then,

$$\overline{W_1}(x) \leq \{1 - \frac{\rho}{c_A(x)}\}[\overline{H}(x)]^c + \frac{\rho}{\rho_A}[\overline{H}(cx)]^{1-\rho}. \tag{3.22}$$

A lower bound similar to (3.20) can be obtained and we omit it here.

Our results can be also applied to many variants of the M/G/1 queue since they are of the form $V * W(x)$, where $V(x)$ is known and $W(x)$ is the equilibrium waiting time distribution of the standard M/G/1 queue(see Neuts, 1986).

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